Favorite-Candidate Voting for Eliminating the Least Popular Candidate in a Metric Space

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Abstract

We study single-candidate voting embedded in a metric space, where both voters and candidates are points in the space, and the distances between voters and candidates specify the voters' preferences over candidates. In the voting, each voter is asked to submit her *favorite* candidate. Given the collection of favorite candidates, a mechanism for eliminating the least popular candidate finds a *committee* containing all candidates but the one to be eliminated.

Each committee is associated with a social value that is the sum of the costs (utilities) it imposes (provides) to the voters. We design mechanisms for finding a committee to optimize the social value. We measure the quality of a mechanism by its *distortion*, defined as the worst-case ratio between the social value of the committee found by the mechanism and the optimal one. We establish new upper and lower bounds on the distortion of mechanisms in this single-candidate voting, for both general metrics and well-motivated special cases.

Keywords: Voting; Social Choice; Distortion

1 Introduction

In social choice theory, a *mechanism* (also referred to as a *voting rule*) aggregates the preferences of multiple voters over a set of candidates, and returns a k-element subset of candidates as a winning committee. An appealing approach to dealing with social choice problems is embedding the input "election" into a *metric space*, *i.e.*, each participant is represented by a point in a metric space, and voters prefer candidates that are closer to them to the ones that are further away. This spatial model has very natural interpretations. For example, in a 2-dimensional Euclidean space, each dimension specifies a political issue (such as military or education), and the position of a voter or candidate identifies the extent to which the individual supports the issues. Recently, this model has attracted attentions from AI researchers, see, *e.g.,* (Anshelevich, Bhardwaj, and Postl 2015; Elkind et al. 2017; Abramowitz, Anshelevich, and Zhu 2019).

The mechanisms in many of the works aforementioned ask each voter for a linear order over candidates. On the other hand, one may note that eliciting so much information on the preferences casts a high burden on the selection rules, and often impairs the privacy of voters. The *simplicity*, which means that each voter is only required to provide a small amount of information, is often a desideratum for good mechanisms. In this paper, we study the *single-candidate vote mechanisms* (named by (Feldman, Fiat, and Golomb 2016)), *scv mechanisms* for short, that ask each voter to cast a vote of a single candidate.

In addition to the top choices of voters, we further assume that the locations of candidates in the metric space are known to the mechanism, while the voters' private locations and numerical preferences are inaccessible, since every political candidate in a typical election should fully announce her opinions on all issues, and thus her location in the space is public information. For example, in the facility location scenario, the city authority, who plans to locate some facilities on a street or a plane, predetermines the potential locations of facilities, based on the landscape, resources and distributions of social communities.

As voters' preferences are specified by their distances to candidates, it is natural to quantify the *quality* of a committee by the associated distances. We evaluate the performance of a mechanism in the standard worst-case analysis benchmark (introduced by Procaccia and Rosenschein (2006)), which defines the *distortion* of a mechanism to be the worst-case ratio between the quality of a committee selected by this mechanism and that of the optimal committee selected by an omniscient mechanism.

Previous work was mainly concerned about the single-winner elections. In this paper, we focus on the antithesis, the multi-winner elections that eliminate the least popular candidate, that is, select a committee containing all candidates but one. These can be regarded as *single-loser* elections, which are well motivated. For example, some enterprises adopt a *last-out* mechanism in the personnel performance appraisal system, which dismisses the employee with the lowest performance in a department each year. Some voting rules in TV talent shows iteratively eliminate one candidate at each time to obtain the final winners.

Our Contributions.

Let m be the number of candidates in the election, and

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W be the winning committee of size $m-1$ selected by a mechanism. We discuss the distortion of mechanisms under two objectives: minimizing the *social cost* and maximizing the *social utility*. In the former case, each voter takes the distance to W (*i.e.*, the smallest distance between her and a candidate in W) as her cost, and the social cost of W is the sum of its distances to all voters. In the latter case, each voter takes her distance to the eliminated candidate (*i.e.*, the one not in W) as her utility, and the social utility of W is the total utility of voters.

In Section 3, we study the distortion of scv mechanisms under the social cost objective. We prove that if the exact locations of the candidates are known, then a simple deterministic mechanism which minimizes the so-called projection distance achieves a distortion of 3, and no deterministic one can do better. In other words, we can compute a 3-approximate solution as long as the input votes are consistent with the true distances, *i.e.,* each vote is indeed a candidate closest to the voter. The most interesting contribution is a randomized scv mechanism with distortion $3-\frac{2}{m}$, which selects each eligible committee with a carefully designed probability. We prove that no randomized mechanism has a distortion better than $3-\frac{2}{m}$, matching the upper bound. The deterministic and randomized mechanisms also satisfy *strategy-proofness*, guaranteeing that each selfish voter always acts truthfully. Moreover, the lower bounds 3 and $3-\frac{2}{m}$ hold even if the voters submit a full preference ranking over all the candidates.

Section 4 focuses on the social utility objective. We show the lower bounds 3 and 1.5 for deterministic and randomized mechanisms, respectively. While the deterministic mechanism that maximizes the projection distance gives a distortion 3 for general metrics, we investigate randomized scv mechanisms for elections in several widely-studied special spaces, *e.g.*, the simplex (where the distance between any two candidates is the same) and the real line (1-Euclidean space). The simplex setting corresponds to the case when candidates share no similarities, i.e., when all candidates are equally different from each other, and the real line is also a well-studied and well-motivated special case.

These results are summarized in Table 1, where LB and UB are shorthands for lower bound and upper bound on the distortion of scv mechanisms.

Table 1: A summary of our results

Objective	Deterministic	Randomized
Social cost	LB: 3 (Prop.3.1) UB: 3 (Thm.3.3)	LB: $3-\frac{2}{m}$ (Prop.3.2) UB: $3-\frac{2}{m}$ (Thm.3.5)
Social utility	LB: 3 (Prop.4.1) UB: 3 (Thm.4.2)	LB: 1.5 (Prop.4.1) UB: $3-\frac{4}{m+2}$ (Simplex, Thm.4.5) 13/7 (Line, Thm.4.6)

In Section 5, we extend our results to a more general setting, where the scv mechanism is required to select a committee of size k, for a predetermined integer $k \leq m-1$. We prove that the simple idea that optimizes the projection distance can achieve a distortion 3 for both the social cost objective and the social utility objective, and no deterministic mechanism can do better. Then we conclude this paper with future research directions.

Related Work.

In social choice theory, Procaccia and Rosenschein (2006) propose a utilitarian approach – the implicit utilitarian voting– by assuming that voters have latent cardinal utilities and report ordinal preferences induced by them. They measure the performance of popular voting rules by the notion of *distortion*. Subsequently, Caragiannis and Procaccia (2011), Oren and Lucier (2014), Boutilier *et al.* (2015), Bhaskar and Ghosh (2018) employ this notion and design selection rules with low distortions.

Anshelevich *et al.* (2015) first embed the election into a metric space, in which the participants are points, and the costs are driven by the distances. They study mechanisms that know only the voters' preference rankings over candidates, but not the underlying metric, and output a single winner. Regarding the objective of minimizing the social cost of the winner, they show the Copeland rule has distortion 5, and prove a lower bound 3 for the distortion of deterministic mechanisms. Later, Skowron and Elkind (2017) show that the class of scoring rules and STV have super-constant distortion. The work of (Goel, Krishnaswamy, and Munagala 2017) proves that the ranking pairs rule has distortion at least 5. Recently, Munagala and Wang (2019) improve the distortion to 4.236, using a weighted tournament rule.

In addition to deterministic rules, randomized rules have also been considered. Random dictatorship that randomly selects the top choice of one of the n voters gets distortion $3-2/n$ (Feldman, Fiat, and Golomb 2016; Anshelevich and Postl 2017). Feldman *et al.* (2016) consider scv mechanisms and strategy-proofness in the metric setting, and propose a 2-distortion mechanism on the real line. The work of (Gross, Anshelevich, and Xia 2017) proposes a very simple mechanism that randomly asks voters for their favorite candidates until two voters agree, achieving low distortion and satisfying some normative properties.

The most related setting to ours appears in (Anshelevich and Zhu 2018), where the candidates' locations are additionally assumed to be known. With this extra location information, they break the best-known upper bound 4.236 mentioned above and present a deterministic 3-distortion scv mechanism for single-winner election.

2 Model

Let $\Omega=(S,d)$ be a metric space, where S is the space and $d: S\times S\rightarrow \mathbb{R}_+$ is the metric. The *distance* between $w\in S$ and $V\subseteq S$ is defined as $d(w,V):=\min_{v\in V} d(w,v)$. Let $N=\{1,\ldots,n\}$ be the set of *voters* (agents), each of whom is located at a private point in S. The location x_i of voter $i\in N$ is her *type*, and the *location profile* of all voters is $\mathbf{x}=(x_1,\ldots,x_n)$. Let $M=\{y_1,\ldots,y_m\}$ be the set of *candidates* (alternatives), each of whom is located at a public point in S. We refer to y_j as the j-th candidate and as her location interchangeably.

The voter prefers the closer candidate, and the nearest candidate is the favorite. Each voter $i\in N$ is asked to submit a single nearest candidate, called her *action* and denoted by $a_i \in M$. The collection of voters' actions is the *action profile* $\mathbf{a} = (a_1, \dots, a_n)$. An *election* in the social choice problem under consideration is a triple $\Gamma = (\Omega, M, \mathbf{a})$. We call a location profile $\mathbf{x}$ *consistent* with election Γ, if each voter's action reveals her real preference, that is, $a_i \in \arg\min_{y \in M} d(x_i, y)$, for every $i \in N$. Denote by $\chi(\Gamma)$ the set of location profiles consistent with Γ.

We are concerned with *mechanisms* that, given an election $\Gamma = (\Omega, M, \mathbf{a})$, select a *committee* (subset of M) of cardinality $m - 1$ as winners. It is assumed that the mechanisms have full information on the metric space Ω and candidate locations M, but they do not know the location profile of voters. Associate each $y \in M$ with the potential committee $M_y := M \setminus \{y\}$. Let $K = \{M_y : y \in M\}$ denote the set of potential committees. A *randomized* mechanism is a function f that maps every action profile $\mathbf{a} \in M^n$ to a random committee $f(\mathbf{a})$ that follows some probability distribution over the potential committees in K. A *deterministic* mechanism f simply selects a specific committee $f(\mathbf{a}) \in K$ with probability 1.

We investigate the performance of mechanisms from the utilitarian perspective, which involves the objectives of minimizing the social cost and maximizing the social utility, respectively.

The social cost objective. Given location profile $\mathbf{x} = (x_i)_{i \in N}$ and committee $Y \in K$, the *cost* of voter $i \in N$ is the distance to the nearest winner, *i.e.,* $d(x_i, Y)$. The *social cost* of Y, denoted as $SC(Y, \mathbf{x})$ or $SC(Y)$ for short, equals $\sum_{i \in N} d(x_i, Y)$. We use $OPT_c(\mathbf{x})$ to denote the social cost of an optimal committee selected by an omniscient mechanism, *i.e.*, $OPT_c(\mathbf{x}) = \min_{Y \in K} SC(Y, \mathbf{x})$. The *distortion* of a (randomized) mechanism f on an election $\Gamma = (\Omega, M, \mathbf{a})$ is

$$dist(f, \Gamma) = \sup_{\mathbf{x} \in \chi(\Gamma)} \frac{\mathbf{E}[SC(f(\mathbf{a}), \mathbf{x})]}{OPT_c(\mathbf{x})}.$$

In other words, it is the worst-case — over the location profiles consistent with Γ — ratio between the expected social cost of the committee selected by the mechanism and the optimal social cost.

The social utility objective. Given location profile $\mathbf{x}$ and committee M_y, the utility of voter i equals the $d(x_i, y)$ to the loser y. The *social utility* of M_y, denoted as $SU(M_y, \mathbf{x})$ or $SU(M_y)$ for short, equals $\sum_{i \in N} d(x_i, y)$. The optimal social utility is $OPT_u(\mathbf{x}) = \max_{Y \in K} SU(Y, \mathbf{x})$, and the *distortion* of a (randomized) mechanism f on election Γ is

$$dist(f, \Gamma) = \sup_{\mathbf{x} \in \chi(\Gamma)} \frac{OPT_u(\mathbf{x})}{\mathbf{E}[SU(f(\mathbf{a}), \mathbf{x})]}.$$

For either of the objectives, we define the *distortion* of a mechanism f as $Dist(f) = \sup_{\Gamma} dist(f, \Gamma)$ by taking the worst case over elections. We call f an *r-distortion mechanism* if $Dist(f) \leq r$.

Strategy-proofness. As in many previous works on social choice, we evaluate the quality of a mechanism under the assumption that the underlying location profile is always consistent with the elections, *i.e.*, the voters act truthfully and submit their nearest candidates. Nevertheless, possibly some voter may use a strategy (that leads to an action and consequently an election with which the location profile may not be consistent) to be better off. A mechanism is *strategy-proof*, if the truth-telling strategy is always optimal for each voter, that is, voting for *any* one of the nearest candidates can *always* optimize her (expected) cost or utility, regardless of the actions of others.

3 Mechanisms for Minimum Social Cost

This section focuses on the objective of minimizing the social cost. We first show the lower bounds on distortion, and propose both deterministic and randomized mechanisms that match the lower bounds.

3.1 Lower Bounds

We prove lower bounds on the distortion of both deterministic and randomized mechanisms by constructing election instances. Our construction is based on the well-known worst case of single-winner election (Anshelevich, Bhardwaj, and Postl 2015; Feldman, Fiat, and Golomb 2016), in which two candidates locate at 0 and 2 on the real line respectively, and each receive a vote. We extend it to our setting by adding $m - 2$ very far candidates, each of whom also receives a vote. Then any mechanism with guaranteed performance must weed out either the candidate locating at 0 or the one at 2; while either option results in a distortion 3.

Proposition 3.1. *For any $m \geq 2$ and the social cost objective, the distortion of any deterministic scv mechanism cannot be smaller than* 3.

Proof. Consider an election Γ in $\mathbb{R}$, where m candidates are located at $y_1 = 0, y_2 = 2, y_3 = L, y_4 = 2L, \dots, y_m = (m - 2)L$ for a large number L, and the action profile of $n = m$ voters is $\mathbf{a} = (0, 2, L, 2L, \dots, (m - 2)L)$.

It is easy to see that any mechanism f with bounded distortion must eliminate either y_1 or y_2. If $y_1 \in f(\mathbf{a})$, then for the location profile $\mathbf{x} = (1, 2, L, 2L, \dots, (m-2)L) \in \chi(\Gamma)$, we have $SC(f(\mathbf{a}), \mathbf{x}) = 3$, and $OPT_c(\mathbf{x}) = 1$ (realized by the optimal committee M_{y_1}), indicating the distortion at least 3. If $y_2 \in f(\mathbf{a})$, the same bound holds for location profile $(0, 1, L, 2L, \dots, (m - 2)L)$. □

Although the example constructed above can provide a lower bound 2 for the distortion of randomized scv mechanism, we prove a better lower bound in the following.

Proposition 3.2. *For the social cost objective, the distortion of any randomized scv mechanism cannot be smaller than $3 - \frac{2}{m}$.*

Proof. Consider an election $\Gamma = (\Omega, M, \mathbf{a})$ with $d(y_i, y_j) = 2$ for any pair of distinct candidates $y_i, y_j \in M$. There are $n = m$ voters, and the action profile is $\mathbf{a} = (y_1, y_2, \dots, y_m)$, that is, each candidate receives a vote from one voter. Since there are in total m potential committees, any randomized mechanism f must select some committee M_y with a probability no more than $\frac{1}{m}$. By symmetry, we can assume w.l.o.g. that $\mathbf{Pr}[f(\mathbf{a}) = M_{y_m}] \leq \frac{1}{m}$.

Now consider the location profile $\mathbf{x} = (y_1, y_2, \dots, y_{m-1}, x_m)$, where the point x_m is at the same distance $d(x_m, y_i) = 1$ from every candidate $y_i \in M$. Obviously, suitable choice of Ω, M and x_m can fulfill all the conditions (*i.e.*, the distances specified satisfy the metric condition), and guarantees that $\mathbf{x}$ is consistent with Γ. (Figure 1 depicts an example for $m = 3$.) The optimal committee is M_{y_m} with optimal social cost $OPT_c(\mathbf{x}) = d(x_m, M_{y_m}) = 1$, while any other committee M_{y_i} with $i \leq m-1$ has a social cost at least $d(x_m, M_{y_i}) + d(y_i, M_{y_i}) = 1 + 2 = 3$. Thus, the expected social cost of the random committee $f(\mathbf{a})$ is $\mathbf{E}[SC(f(\mathbf{a}), \mathbf{x})] \geq \frac{1}{m} OPT_c(\mathbf{x}) + (1 - \frac{1}{m})3 = 3 - \frac{2}{m}$, showing that the distortion of f is at least $3 - \frac{2}{m}$. $\square$

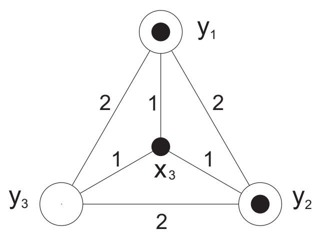

Figure 1: Three candidates are indicated by hollow circles, three voters are indicated by solid disks. Each candidate receives a vote. The numbers near edges indicate the distances, which are not Euclidean. The point x_3 is at distance 1 from every candidate, and the optimal solution eliminates y_3.

It is worth pointing out that the two election examples constructed in the proofs of Propositions 3.1 and 3.2 can be applied to the election that asks each voter to submit a preference ranking. Thus the lower bounds in these two proportions also hold for the mechanisms that aggregate voters' rankings over candidates.

3.2 Projection Mechanism

Given an action profile $\mathbf{a} = (a_i)_{i \in N}$, it can be viewed as a projection of the location profile of voters to the location set of candidates. For any subset $W \subseteq M$, we define its *projection distance* w.r.t. $\mathbf{a}$ as $pd_{\mathbf{a}}(W) := \sum_{i \in N} d(a_i, W)$. In the remainder of this paper, we use $d(i, V)$ instead of $d(x_i, V)$ for $i \in N$ and $V \subseteq S$, when the context is clear. Now we are ready to present a deterministic mechanism which ensures distortion 3 matching the lower bound in Proposition 3.1, by selecting a committee that minimizes the projection distance.

Mechanism 1 (MIN-PROJECTION-DISTANCE)**.** Given an election $\Gamma = (\Omega, M, \mathbf{a})$, mechanism f deterministically outputs a committee $f(\mathbf{a})$ with the smallest projection distance, that is, $f(\mathbf{a}) \in \text{argmin}_{M_y : y \in M} pd_{\mathbf{a}}(M_y)$; ties are broken arbitrarily.

The spirit of this mechanism is treating the action of each voter as her location.

Theorem 3.3. MIN-PROJECTION-DISTANCE *is a deterministic, strategy-proof, polynomial-time and* 3*-distortion scv mechanism for the social cost objective.*

Proof. The polynomial-time computability is straightforward since the number of possible committees is $|K| = m$.

For the strategy-proofness, we show that the truth-telling strategy always gives each voter a minimum cost. Suppose action a_i is a nearest candidate of voter i, and $a'_i \in M \setminus \{a_i\}$ is another arbitrary action. Given the actions $\mathbf{a}_{-i}$ of other voters, consider the action profiles $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ and $\mathbf{a}' = (a'_i, \mathbf{a}_{-i})$. The output of the mechanism is $f(\mathbf{a}) = Y \in \text{argmin}_{M_y} pd_{\mathbf{a}}(M_y)$ and $f(\mathbf{a}') = Y' \in \text{argmin}_{M_y} pd_{\mathbf{a}'}(M_y)$. We only need to consider the case where $Y \neq Y'$. If $a_i \in Y$, the cost of voter i is minimized when she tells the truth. So we assume $a_i \notin Y$, which along with $k = m - 1$ implies $a_i \in Y'$. If $a'_i \in Y'$, the projection distance of Y on $\mathbf{a}'$ is $pd_{\mathbf{a}'}(Y) = pd_{\mathbf{a}}(Y) - d(a_i, Y) < pd_{\mathbf{a}}(Y) \leq pd_{\mathbf{a}}(Y')$, and the projection distance of Y' on $\mathbf{a}'$ is $pd_{\mathbf{a}'}(Y') = pd_{\mathbf{a}}(Y')$ since both a_i and a'_i are in Y'. So we have $pd_{\mathbf{a}'}(Y) < pd_{\mathbf{a}'}(Y')$, which contradicts the selection rule of the mechanism, and reduces to the case of $a_i \in Y' \setminus Y$ and $a'_i \in Y \setminus Y'$. Now we have $pd_{\mathbf{a}'}(Y') > pd_{\mathbf{a}}(Y') \geq pd_{\mathbf{a}}(Y) > pd_{\mathbf{a}'}(Y)$, which also contradicts the selection rule. Therefore, voter i's cost when reporting a_i is always no more than her cost when reporting any a'_i, which proves the strategy-proofness.

Given election Γ and any consistent location profile $\mathbf{x} = (x_i)_{i \in N} \in \chi(\Gamma)$, let Y^* be the optimal committee, and Y be the output by the mechanism. Then

$$\begin{aligned}\frac{SC(Y)}{SC(Y^*)} &= \frac{\sum_{i \in N} d(x_i, Y)}{\sum_{i \in N} d(x_i, Y^*)} \\ &\leq \frac{\sum_{i \in N} d(x_i, a_i)}{\sum_{i \in N} d(x_i, Y^*)} + \frac{\sum_{i \in N} d(a_i, Y)}{\sum_{i \in N} d(x_i, Y^*)} \\ &\leq 1 + \frac{\sum_{i \in N} d(a_i, Y)}{\sum_{i \in N} d(x_i, Y^*)}.\end{aligned}$$

For every $i \in N$, recalling from the consistency that $d(x_i, a_i) = \min_{y \in M} d(x_i, y) \leq d(x_i, Y^*)$, we have $2d(x_i, Y^*) \geq d(x_i, Y^*) + d(x_i, a_i) \geq d(a_i, Y^*)$. Therefore

$$\frac{SC(Y)}{SC(Y^*)} \leq 1 + \frac{\sum_{i \in N} d(a_i, Y)}{\frac{1}{2}\sum_{i \in N} d(a_i, Y^*)} = 1 + \frac{2pd_{\mathbf{a}}(Y)}{pd_{\mathbf{a}}(Y^*)} \leq 3,$$

where the last inequality is guaranteed by the selection rule of the mechanism. $\square$

3.3 Power-Proportionality Mechanism

Inspired by (Anshelevich and Postl 2017), we establish in the following, for any given randomized scv mechanism and location profile, an upper bound on the ratio between the expected social cost of the committee selected by the mechanism, and the optimal social cost. With the help of this upper bound, we design a randomized scv mechanism, and prove its strategy-proofness and distortion (which matches the lower bound in Proposition 3.2).

Before presenting the formal description of the upper bound, we make a partition of the voter set according to voters' actions. Given an action profile $\mathbf{a} = (a_i)_{i \in N}$, for each candidate $y \in M$, let $N_{\mathbf{a},y} = \{i \in N | a_i = y\}$ denote the subset of voters whose actions are y. Then $(N_{\mathbf{a},y})_{y \in M}$ forms a partition of N.

Lemma 3.4. *Given a randomized scv mechanism and an election $\Gamma = (\Omega, M, \mathbf{a})$, suppose the probability that the mechanism selects each $M_y \in K$ as winners is $P(M_y)$. Then, for any location profile $\mathbf{x} \in \chi(\Gamma)$ and any optimal committee M_{y^*} with $y^* \in M$, the following holds:*

$$\frac{\sum_{y\in M} P(M_y)SC(M_y)}{SC(M_{y^*})} \le 1 + \frac{2\sum_{y\neq y^*} P(M_y)|N_{\mathbf{a},y}|d(y, M_y)}{|N_{\mathbf{a},y^*}|d(y^*, M_{y^*})}. \tag{1}$$

Proof. For each voter $i \in N$, note that $i \in N_{\mathbf{a},a_i}$ and (from the consistency of $\mathbf{x}$) that a_i is a nearest candidate for i. If $y \in M \setminus \{a_i\}$, *i.e.*, $i \in N \setminus N_{\mathbf{a},y}$, then $a_i \in M_y$. For every committee $M_y \neq M_{y^*}$, notice that the candidate y belongs to M_{y^*}, giving $d(y, M_{y^*}) = 0$. So, the social cost of M_y with $y \neq y^*$ is upper bounded by

$$\begin{aligned}
& SC(M_y, \mathbf{x}) \\
= & \sum_{i\in N\setminus N_{\mathbf{a},y}} d(x_i, M_y) + \sum_{i\in N_{\mathbf{a},y}} d(x_i, M_y) \\
\le & \sum_{i\in N\setminus N_{\mathbf{a},y}} d(x_i, M_{y^*}) + \sum_{i\in N_{\mathbf{a},y}} (d(x_i, y) + d(y, M_y)) \\
= & SC(M_{y^*}, \mathbf{x}) + |N_{\mathbf{a},y}|d(y, M_y).
\end{aligned}$$

Since $2d(x_i, M_{y^*}) \ge d(x_i, a_i) + d(x_i, M_{y^*}) \ge d(a_i, M_{y^*})$ for every $i \in N$, the optimal social cost is lower bounded by

$$\begin{aligned}
SC(M_{y^*}, \mathbf{x}) &= \sum_{i\in N} d(x_i, M_{y^*}) \\
&\ge \sum_{i\in N} \frac{d(a_i, M_{y^*})}{2} \\
&= \frac{1}{2} \sum_{y\in M} |N_{\mathbf{a},y}|d(y, M_{y^*}) \\
&= \frac{1}{2} |N_{\mathbf{a},y^*}|d(y^*, M_{y^*}).
\end{aligned}$$

The above two bounds give the following estimate on the ratio of the expected social cost of the committee output by the mechanism to the optimum:

$$\begin{aligned}
& \frac{\sum_{y\in M} P(M_y)SC(M_y,\mathbf{x})}{SC(M_{y^*},\mathbf{x})} \\
= & P(M_{y^*}) + \frac{\sum_{y\in M\setminus\{y^*\}} P(M_y)SC(M_y,\mathbf{x})}{SC(M_{y^*},\mathbf{x})} \\
\le & P(M_{y^*}) + \frac{\sum_{y\in M\setminus\{y^*\}} P(M_y)(SC(M_{y^*},\mathbf{x})+|N_{\mathbf{a},y}|d(y,M_y))}{SC(M_{y^*},\mathbf{x})} \\
\le & 1 + \frac{2\sum_{y\in M\setminus\{y^*\}} P(M_y)|N_{\mathbf{a},y}|d(y,M_y)}{|N_{\mathbf{a},y^*}|d(y^*,M_{y^*})},
\end{aligned}$$

which proves the lemma. □

A natural idea to design a mechanism is making the right hand side of inequality (1) as small as possible. Next, we seek a suitable mechanism whose probabilities of winning set selections achieve this goal.

Mechanism 2 (POWER-PROPORTIONALITY)**.** Given an election $\Gamma = (\Omega, M, \mathbf{a})$, for every committee $M_y \in K$, the winning probability is

$$P(M_y) = \frac{|N_{\mathbf{a},y}|^{-m} d(y, M_y)^{-m}}{\sum_{z\in M} |N_{\mathbf{a},z}|^{-m} d(z, M_z)^{-m}}. \tag{2}$$

Theorem 3.5. POWER-PROPORTIONALITY *is a randomized scv mechanism that is strategy-proof and has distortion at most $3 - \frac{2}{m}$ for social cost objective.*

Proof. As $\sum_{y\in M} P(M_y) = 1$, the probability distribution is well-defined. To see the strategy-proofness, consider any location profile $\mathbf{x}$ and an arbitrary voter i, one of whose nearest candidates being y. It is easy to see that, i voting for y (in comparison with not doing so) increases the size of $N_{\mathbf{a},y}$, and decreases the probability $P(M_y)$. The expected cost of voter i is $P(M_y)d(x_i, M_y) + (1 - P(M_y))d(x_i, y)$. Since $d(x_i, y) \le d(x_i, M_y)$, the truth-telling strategy always minimizes her expected cost, which indicates the strategy-proofness.

Next, we investigate the distortion w.r.t. $\mathbf{x} \in \chi(\Gamma)$. By Lemma 3.4, substituting the probability (2) into inequality (1), we have

$$\frac{\sum_{y\in M} P(M_y)SC(M_y, \mathbf{x})}{SC(M_{y^*}, \mathbf{x})} \le 1 + \frac{2\sum_{y\in M\setminus\{y^*\}} |N_{\mathbf{a},y}|^{1-m} d(y, M_y)^{1-m}}{|N_{\mathbf{a},y^*}|d(y^*, M_{y^*}) \sum_{y\in M} |N_{\mathbf{a},y}|^{-m} d(y, M_y)^{-m}}. \tag{3}$$

Now we compute the maximum value of the right hand side in (3) by the function $g : \mathbb{R}_+^m \to \mathbb{R}$,

$$g(\alpha_1, \ldots, \alpha_m) = 1 + \frac{2\alpha_1 \sum_{i=2}^m \alpha_i^{m-1}}{\sum_{i=1}^m \alpha_i^m}.$$

By the derivative of this function, we know that the maximum value is attained when $\alpha_1 = \cdots = \alpha_m$, that is, $\max g(\alpha_1, \ldots, \alpha_m) = g(\alpha_1, \ldots, \alpha_1) = 3 - \frac{2}{m}$. The right hand side of (3) has the same form as g, and it is also at most $3 - \frac{2}{m}$, which gives the upper bound of the distortion. □

4 Mechanisms for Maximum Social Utility

In this section, we focus on the social utility objective. Each voter targets a favorite candidate, and takes the distance to the eliminated candidate as her utility, as she wants to stay as far away from the nuisance as possible.

By a simple adaptation to the proof of Proposition 3.1, one can easily obtain the following lower bounds for both deterministic and randomized mechanisms.

Proposition 4.1. *For the social utility objective, no deterministic (resp. randomized) scv mechanism can have a distortion smaller than* 3 *(resp.* 1.5*).*

We present in Section 4.1 a deterministic svc mechanism with distortion 3, using a dual idea of Mechanism 1. Then, we provide in Sections 4.2 and 4.3 randomized mechanisms for some important special metric spaces.

4.1 Projection Mechanism

Recall that the *projection distance* of candidate $y \in M$ on an action profile $\mathbf{a} = (a_i)_{i\in N}$ is $pd_{\mathbf{a}}(y) = \sum_{i\in N} d(a_i, y)$. We follow the dual spirit of Mechanism 1 to select a committee with the eliminated candidate maximizing the projection distance.

Mechanism 3 (MAX-PROJECTION-DISTANCE). Given an election $\Gamma = (\Omega, M, \mathbf{a})$, the deterministic mechanism f outputs committee M_y where y has the largest projection distance on $\mathbf{a}$, that is, $y \in \arg\max_{w \in M} pd_{\mathbf{a}}(w)$ and $f(\mathbf{a}) = M_y$, breaking ties arbitrarily.

The following 3-distortion performance guarantee can be proved by an argument that is completely symmetrical with the proof of Theorem 3.3.

Theorem 4.2. MAX-PROJECTION-DISTANCE *is a deterministic polynomial-time* 3*-distortion scv mechanism for the social utility objective.*

This 3-distortion scv mechanism is the best that one can expect for deterministic mechanisms, in view of Proposition 4.1. In contrast to Mechanism 1, it is not strategy-proof: When a voter has two favorite candidates and votes for them respectively, resulting in different action profiles, the corresponding outputs of MAX-PROJECTION-DISTANCE may be two candidates that have different distances to her. Therefore, to maximize her utility, she has to vote for the specific candidate who leads to a better outcome.

4.2 Proportionality Mechanism

A natural idea for randomization is selecting a committee in K with a probability proportional to the number of voters who vote for it. We show the strategy-proofness, and evaluate the distortion in the two-candidate case and simplex case.

Recall that $N_{\mathbf{a},y} = \{i \in N | a_i = y\}$ is the set of voters who vote for the candidate $y \in M$.

Mechanism 4 (PROPORTIONALITY)**.** Given an election $\Gamma = (\Omega, M, \mathbf{a})$, for each committee M_y, $y \in M$, the winning probability is

$$P(M_y) = \frac{n - |N_{\mathbf{a},y}|}{(m-1)n}.$$

Note that the probability distribution is well-defined, as the sum of $n - |N_{\mathbf{a},y}|$ over $y \in M$ is $(m-1)n$.

Lemma 4.3. PROPORTIONALITY *is strategy-proof.*

Proof. Consider an arbitrary voter $i \in N$, and suppose $y \in M$ is her favorite candidate. If voter i switches her action from y to any other $y' \in M_y$, then the probability $P(M_y)$ increases, $P(M_{y'})$ decreases, and all other probabilities remain the same. The expected utility of voter i is $P(M_y)d(x_i, y) + P(M_{y'})d(x_i, y') + U$ with a fixed value U. Since $d(x_i, y) \le d(x_i, y')$, this implies that the expected utility is non-increasing by switching from y to y'. Therefore, being truthful is the optimal strategy, regardless of the actions of other voters. □

By an analysis similar to the proof of Lemma 3.4, we obtain a lower bound on the ratio between the expected social utility of the selection and the optimal utility.

Lemma 4.4. *Given a single-winner election* $\Gamma = (\Omega, M, \mathbf{a})$ *and location profile* $\mathbf{x} \in \chi(\Gamma)$*, suppose* M_{y^*} *is an optimal committee. For any randomized mechanism that selects* M_y $(y \in M)$ *as winning committee with probability* $P(M_y)$*, the expected social utility satisfies*

$$\begin{aligned}
&\frac{\sum_{y \in M} P(M_y) SU(M_y)}{SU(M_{y^*})} \\
\ge\ & 1 - \sum_{y \in M_{y^*}} P(M_y) \left(1 + \frac{\sum_{z \in M} |N_{\mathbf{a},z}| d(z,y)}{2(n - |N_{\mathbf{a},y^*}|) d(y, y^*)}\right)^{-1}.
\end{aligned}$$

With the help of Lemma 4.4, we can upper bound the distortion of PROPORTIONALITY in the 2-candidate case (*i.e.*, $m = 2$) and simplex case. We say the candidates form a *simplex*, if the distance between any two candidates is the same, say 2, *i.e.*, $d(y, z) = 2$ for all distinct $y, z \in M$.[1]

Theorem 4.5. *For the social utility objective,* PROPORTIONALITY *has distortion*
(i) at most 1.523 *when* $m = 2$*;*
(ii) at most $3 - \frac{4}{m+2}$ *when candidates form a simplex.*

Proof. (i) For any election Γ and consistent location profile $\mathbf{x} \in \chi(\Gamma)$, suppose y is the optimal candidate (singleton committee), and y^* is the other one. By Lemma 4.4, we have

$$\begin{aligned}
&\frac{P(y)SU(y) + P(y^*)SU(y^*)}{SU(y)} \\
\ge\ & 1 - P(y^*) \left(1 + \frac{|N_{\mathbf{a},y^*}|}{2(n - |N_{\mathbf{a},y^*}|)}\right)^{-1} \\
=\ & 1 - \frac{|N_{\mathbf{a},y^*}|}{n} \left(1 + \frac{|N_{\mathbf{a},y^*}|}{2(n - |N_{\mathbf{a},y^*}|)}\right)^{-1} \\
=\ & 1 - \left(\frac{n}{|N_{\mathbf{a},y^*}|} + \frac{n}{2(n - |N_{\mathbf{a},y^*}|)}\right)^{-1} \\
\ge\ & 1 - \left(1.5 + \sqrt{2}\right)^{-1}
\end{aligned}$$

Therefore, the distortion is at most $(1 - (1.5 + \sqrt{2})^{-1})^{-1} = (5 + 4\sqrt{2})/7 = 1.5224...$

(ii) can be proved in a similar but more involved analysis, which is relegated to Supplementary Material. □

4.3 Mechanisms on the Real Line

We now consider the case where all voters and candidates are located on the real line, and the metric is defined as the Euclidean distance. This setting simulates the scenario in which an authority wants to build a facility on a street, and has been extensively studied for obnoxious facility games. The results of (Cheng, Yu, and Zhang 2011) implies that an optimal committee must eliminate one of the two endpoints of the line segment spanned by $y_1, \ldots, y_m$. This nice fact directly provides a randomized strategy-proof 2-distortion mechanism that eliminates the leftmost candidate and the rightmost candidate with probability $\frac{1}{2}$, respectively. Next, we improve the distortion by a more involved probability distribution of selection, at a cost of losing the strategy-proofness.

[1]The simplex is studied in (Anshelevich and Postl 2017) for single-winner election, under some additional assumption on distances.

Mechanism 5 (LEFT-OR-RIGHT)**.** Given an election $\Gamma = (\mathbb{R}, M, \mathbf{a})$, where the leftmost and rightmost candidate are located at $y_1 = 0$ and $y_m = L$, respectively. Denote by n_1, n_2 the number of voters whose actions are on $[0, \frac{L}{2}], (\frac{L}{2}, L]$, respectively. Select M_{y_i} with probability $P(M_{y_i}), i = 1, m$, as specified below:

- If $n_1 > n_2$, then $P(M_{y_1}) = \frac{6}{13}$ and $P(M_{y_m}) = \frac{7}{13}$.
- If $n_1 < n_2$, then $P(M_{y_1}) = \frac{7}{13}$ and $P(M_{y_m}) = \frac{6}{13}$.
- If $n_1 = n_2$, then $P(M_{y_1}) = P(M_{y_m}) = \frac{1}{2}$.

Theorem 4.6. LEFT-OR-RIGHT *is a randomized* $\frac{13}{7}$*-distortion scv mechanism for the social utility objective.*

Proof. For any election $\Gamma = (\mathbb{R}, M, \mathbf{a})$ and consistent location profile $\mathbf{x} \in \chi(\Gamma)$, we show that the performance ratio $\frac{OPT_u(\mathbf{x})}{\mathbf{E}[SU(f(\mathbf{a}),\mathbf{x})]}$ is upper bounded by $\frac{13}{7}$, where f denotes mechanism LEFT-OR-RIGHT. It is easy to see that the worst case w.r.t. the performance ratio must occur when all voters are also located on interval $[0, L]$. (If some x_i is smaller than 0 or larger than L, then changing it to 0 or L would not decrease the ratio.) So we assume that $x_i \in [0, L]$ for all $i \in N$, and only consider the line segment $[0, L]$.

If $n_1 = n_2$, then the expected social utility of the outcome is

$$\begin{aligned}\mathbf{E}[SU(f(\mathbf{a}))] &= \tfrac{1}{2}SU(M_{y_1}) + \tfrac{1}{2}SU(M_{y_m}) \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^n x_i + \tfrac{1}{2}\sum_{i=1}^n (L - x_i) = \tfrac{Ln}{2},\end{aligned}$$

and the optimal social utility is $OPT_u(\mathbf{x}) = \max\{SU(M_{y_1}), SU(M_{y_m})\}$. Since $n_1 = n_2$, we have

$$SU(M_{y_1}) = \textstyle\sum_{i=1}^n x_i \leq \tfrac{3}{4}Ln_1 + Ln_2 = \tfrac{7Ln}{8},$$

where $SU(M_{y_1})$ reaches the upper bound when n_1 voters who vote for the midpoint candidate $\frac{L}{2}$ are located at $\frac{3L}{4}$, and n_2 voters who vote for $y_m = L$ are located at L. Similarly, we have

$$SU(M_{y_m}) \leq \tfrac{7Ln}{8}.$$

Therefore, we have $OPT_u(\mathbf{x}) \leq \frac{7}{4}\mathbf{E}[SU(f(\mathbf{a}))] < \frac{13}{7}\mathbf{E}[SU(f(\mathbf{a}))]$ as desired.

When $n_1 \neq n_2$, by symmetry, we only discuss the case $n_1 > n_2$. Recall that an optimal solution eliminates either y_1 or y_m. First, if the optimal committee is M_{y_1}, we also have $OPT_u(\mathbf{x}) = \sum_{i=1}^n x_i \leq \frac{3}{4}Ln_1 + Ln_2$ by the same reasoning as above. In turn, $\frac{3}{4}n_1 + n_2 = n - \frac{n_1}{4} < \frac{7}{8}n$ gives $OPT_u(\mathbf{x}) < \frac{7Ln}{8}$. It follows that

$$\begin{aligned}\mathbf{E}[SU(f(\mathbf{a}))] &= \tfrac{6}{13}\textstyle\sum_{i=1}^n x_i + \tfrac{7}{13}\sum_{i=1}^n (L - x_i) \\ &= \tfrac{7}{13}Ln - \tfrac{1}{13}\textstyle\sum_{i=1}^n x_i \\ &> \tfrac{8}{13}OPT_u(\mathbf{x}) - \tfrac{1}{13}OPT_u(\mathbf{x}) \\ &= \tfrac{7}{13}OPT_u(\mathbf{x}).\end{aligned}$$

Next, if the optimal committee is M_{y_m}, with $\epsilon > 0$ being infinitesimal, we have

$$OPT_u(\mathbf{x}) = \textstyle\sum_{i=1}^n (L - x_i) \leq Ln_1 + \tfrac{3}{4}Ln_2 - \epsilon < Ln,$$

where the first inequality holds with equality when n_2 voters who vote for $\frac{L}{2} + \epsilon'$ are located at $\frac{L}{4} + \epsilon'$ ($\epsilon' > 0$ being infinitesimal), and n_1 voters who vote for $y_1 = 0$ are located at 0. Therefore,

$$\begin{aligned}\mathbf{E}[SU(f(\mathbf{a}))] &= \tfrac{6}{13}\textstyle\sum_{i=1}^n x_i + \tfrac{7}{13}\sum_{i=1}^n (L - x_i) \\ &= \tfrac{6}{13}Ln + \tfrac{1}{13}\textstyle\sum_{i=1}^n (L - x_i) \\ &> \tfrac{6}{13}OPT_u(\mathbf{x}) + \tfrac{1}{13}OPT_u(\mathbf{x}) \\ &= \tfrac{7}{13}OPT_u(\mathbf{x}).\end{aligned}$$

The proof is complete. □

5 Concluding Remarks

In this paper we are concerned with the *scv* mechanisms for single-loser election, instead of *ranking* mechanisms that ask the ordinal preferences of voters. We study how well, in terms of minimizing (maximizing) social cost (utility), the mechanisms that only receive the information on top-ranked candidates can compete with omniscient selection rules. From the worst-case perspective, our results show that accessing the very limited information is often enough, in view that the performance guarantees of the mechanisms we propose match the lower bounds which hold even when ranking preferences are known.

Extension. The good performances of scv mechanisms can be extended to a more general task: selecting a size-k committee W as winners for a predetermined integer $k \leq m - 1$. The voters may take the distance to the winners' set W as their costs, or take the distance to the losers' set $M\backslash W$ as their utilities. For the social cost (utility) objective, a couple of ideas and results presented in Sections 3 and 4 can be generalized. Specifically, we obtain the following lower bounds (LB) and upper bounds (UB) on the distortions of scv mechanisms for selecting a size-k committee:

Objective	Deterministic	Randomized
Social cost	LB = UB = 3	LB: 2
Social utility	LB = UB = 3	LB: 1.5

The upper bound 3 for the social cost (utility) objective is guaranteed by outputting a size-k committee that minimizes the projection distance (whose complement set maximizes the projection distance). More details could be found in Supplementary Material.

Future direction. Although strategy-proof mechanisms for the single-winner and single-loser voting have been explored more or less, for the general problem of selecting a size-k committee by scv rules, so far, to the best of our knowledge, there is no performance-guaranteed mechanism that is strategy-proof, even for $k = 2$. This suggests an interesting research direction for scv mechanism design. Except for the proportional idea employed by Mechanism 2 and 4, the quadratic proportionality (Meir, Procaccia, and Rosenschein 2012; Anshelevich and Postl 2017) or other proportional probabilities relying on k may be useful.

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