Characterizability in Belief Revision

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Abstract

A formal framework is given for the postulate characterizability of a class of belief revision operators, obtained from a class of partial preorders using minimization. It is shown that for classes of posets characterizability is equivalent to a special kind of definability in monadic second-order logic, which turns out to be incomparable to first-order definability. Several examples are given of characterizable and non-characterizable classes. For example, it is shown that the class of revision operators obtained from posets which are not total is not characterizable.

Introduction

The main approach to belief change is the AGM approach pioneered by (Alchourrón, Gärdenfors, and Makinson 1985). It provides many characterization results for belief change operators in terms of rationality postulates (Hansson 1999).

Are there cases where no characterization can be given? It seems to be of interest to consider such questions in logical frameworks attempting to formalize aspects of human reasoning, as a means to better understand their expressivity and other related issues. Similar question were studied, for example, in modal logic (Blackburn, de Rijke, and Venema 2001) and dynamic epistemic logic (van Ditmarsch, van der Hoek, and Kooi 2007).

Answering this non-axiomatizability question presupposes a formal definition of a postulate. However, as noted in the survey paper (Fermé and Hansson 2011)

> "theories of belief change developed in the AGM tradition are not logics in a strict sense, but rather informal axiomatic theories of belief change. Instead of characterizing the models of belief and belief change in a formalized object language, the AGM approach uses a natural language (ordinary mathematical English) to characterize the mathematical structures under study."

As far as we know, the first work on characterizability in belief revision, including a formal definition of characterizability and a non-characterizability result, is by Schlechta in the framework of *distance semantics*, forming part of his general study of nonmonotonic logic(Schlechta 2004; Lehmann, Magidor, and Schlechta 2001). Schlechta's work is extended in (Ben-Naim 2006).

In this paper, we provide a formal framework for studying characterizability based on the approach of Katsuno and Mendelzon (Katsuno and Mendelzon 1991). A revision operator $*$ is considered to assign a revised knowledge base $K * \varphi$ to every knowledge base K and every revising formula φ. However, the results remain valid if one considers $*$ to act on a fixed knowledge base K and an arbitrary revising formula φ. We are not discussing iterated revision here, so there is no interaction between the revisions of different knowledge bases. Katsuno and Mendelzon prove the following results:

a) There is a finite set of postulates such that a revision operator satisfies these postulates iff there is a faithful *total preorder* representing it with minimization.

*b)*There is a finite set of postulates such that a revision operator satisfies these postulates iff there is a faithful *partial preorder* representing it with minimization.

*c)*There is a finite set of postulates such that a revision operator satisfies these postulates iff there is a faithful *poset* representing it with minimization.

Part *a)* is a finite version of Grove's characterization of the AGM postulates in terms of systems of spheres (Grove 1988). The postulates for parts *b)* and *c)* are the same, and different from *a)*.

We consider the following general question.

Problem 1. *Let $\mathcal{R}$ be a family of partial preorders. Is there a finite set of postulates such that a revision operator satisfies these postulates iff there is a faithful* partial preorder from $\mathcal{R}$ *representing it with minimization?*

This problem corresponds to situations when the partial preorders over the possible worlds are known to satisfy certain properties and the question is what additional properties of the revision operators are entailed by this extra information. For example, total orders mean that any two models can be compared. Bounded height means that the length of any chain of comparable models is bounded, thus, in a sense, our ability to distinguish models is limited. Another interesting class (not considered in this paper) is that of orders of bounded dimension (Trotter 1992). Here the assumption is that the comparison between two models is determined by

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a bounded number of underlying criteria, each represented by a total order, and a model is preferred to another if it is preferred according to each criterion.

Our goal is to "characterize postulate characterizability". This is achieved for families of partial *orders* by showing that the answer to the question is positive iff the family is definable by a $\forall MSO_{\min}$ sentence, which is a special kind of monadic second-order sentence. Interestingly, $\forall MSO_{\min}$ definability turns out to be incomparable to first-order definability. We use the characterization to give several examples of characterizable and non-characterizable classes of revision operators. The negative results are proved using a "forgetful" version of the Ajtai-Fagin variant of Ehrenfeucht-Fraïssé games.

In Sections 2-7 of the paper we discuss the framework for the discussion of characterizability and provide the tools used in the second half of the paper. Sections 8-10 define the classes of revision operators considered, and prove the results on characterizability, resp., non-characterizability. Section 11 contains a diagram summarizing the relationships between characterizability and logical definability. Due to lack of space, proofs are outlined only or omitted.

Preliminaries

We consider propositional logic knowledge bases K over a fixed finite set of variables. We write K_n to indicate that K is over n variables. Truth assignments (or interpretations) are assignments of truth values to the variables. The set of truth assignments satisfying a formula φ is denoted by $|\varphi|$. Given a set A of truth assignments, $\langle A\rangle$ is some formula φ such that $|\varphi| = A$. A knowledge base is represented by a single formula [1].

Given a knowledge base K, a belief revision operator $*$ assigns a formula $K*\varphi$ to every formula φ. Here φ is called the revising formula, and $K*\varphi$ is called the revised knowledge base.

A partial preorder is $R = (X, \leq)$, where X is a finite ground set and $\leq$ is a reflexive, transitive binary relation. A poset (or partial order) is, in addition, antisymmetric. We write $a \sim b$ if a and b are incomparable. The comparability graph of R is the undirected graph over X such that for any pair of vertices (a, b) is an edge iff $a \leq b$ or $b \leq a$. An element a is minimal if there is no b such that $b < a$, where $b < a$ iff $b \leq a$ but $a \not\leq b$. If $X' \subseteq X$ then a is minimal in X' if $a \in X'$ and there is no $b \in X'$ such that $b < a$. The set of minimal elements of X' is denoted by $\min_{\leq} X'$.

Definition 0.1. *(Faithful partial preorder) A* faithful partial preorder *for a knowledge base K_n is a pair $F = (R, t)$, where $R = (X, \leq)$ is a partial preorder on 2^n elements and $t : X \to \{0,1\}^n$ is a bijection between the elements of X and truth assignments, such that*

1. *$a \in X$ is minimal iff $t(a)$ satisfies K_n,*
2. *if $t(a)$ satisfies K_n and $t(b)$ falsifies K_n then $a < b$.*

[1] We note that because of finiteness this representation corresponds to the belief set framework. Computational complexity issues are not discussed here thus the details of the representation are irrelevant.

In the standard definition the partial preorder is defined over the set of truth assignments. For our discussion it is more convenient to separate the partial preorder and the labeling of its elements by truth assignments. Similar distinctions are made in modal logic as well (Blackburn, de Rijke, and Venema 2001).

Definition 0.2. *(Revision using minimization) The* revision operator $*_F$ for K, determined by a faithful partial preorder F for K, using minimization *is*

$$K *_F \varphi = \langle \min_{\leq} t^{-1}(|\varphi|)\rangle.$$

Thus the revised knowledge base is satisfied by the minimal satisfying truth assignments of the revising formula. Faithfulness implies that if the revising formula is consistent with the knowledge base then the revised knowledge base is the conjunction of the knowledge base and the revising formula.

We will use some notions from finite model theory. General introductions to the topic are given in (Ebbinghaus and Flum 2006; Libkin 2004). The notions used are introduced in the later sections, so our discussion is essentially self-contained.

Postulates

Consider the AGM postulate

$$\text{if } K \wedge \varphi \text{ is satisfiable, then } K * \phi = K \wedge \varphi.$$

Here $K, K*\varphi, \varphi$ and ψ can be considered as unary predicates over the set of interpretations, and thus the above postulate can be rewritten as

$$[\exists x(K(x) \wedge \varphi(x)] \to [\forall x\big((K*\varphi)(x) \leftrightarrow (K(x) \wedge \varphi(x))\big)]. \quad (1)$$

Postulates refer to a fixed knowledge base K, and are implicitly universally quantified over formula symbols such as φ, ψ. They express general requirements that are supposed to hold for all revising formulas. Generalizing these examples, a postulate is defined as follows.

Definition 0.3. *(Postulate) A* postulate $\mathcal{P}$ *is a first-order sentence with unary predicate symbols $K, \varphi_1, \ldots, \varphi_\ell$ and $K*\mu_1, \ldots, K*\mu_m$, where $\mu_1, \ldots, \mu_m$ are Boolean combinations of $\varphi_1, \ldots, \varphi_\ell$.*

A revision operator satisfies a postulate for a knowledge base K if the postulate holds for all $\varphi_1, \ldots, \varphi_\ell$, with the variables ranging over the set of truth assignments.

This definition covers most postulates in (Katsuno and Mendelzon 1991) and in Section 7.3 of (Hansson 1999). The postulate of relevance seems to be an example which is not covered by this definition. As we are mainly interested in negative results, we give a quite general definition of postulates, as this makes the negative results stronger. One could consider other notions for the syntax of the sentences, in order to distinguish between the expressive power of different types of postulates.

Translation

We define a translation of postulates as defined in Definition 0.3 into sentences over an extension of the language of partial preorders. The language of partial preorders contains a binary relation symbol $\leq$ and equality.

The translated sentences also contain additional unary predicate symbols $A_1, \ldots, A_\ell$. These correspond to propositional formulas $\varphi_1, \ldots, \varphi_\ell$ occurring in the postulates. Given a Boolean combination μ of $\varphi_1, \ldots, \varphi_\ell$, we denote by $\hat{\mu}$ the first-order formula obtained by replacing the φ's with A's. For instance, for $\mu(x) = \varphi_1(x) \wedge \varphi_2(x)$ one has $\hat{\mu}(x) = A_1(x) \wedge A_2(x)$.

Given a formula ν over the language $\leq, A_1, \ldots, A_\ell$ with a single free variable x we write $\min_{\leq}^{\nu}$ for a formula expressing that x is a minimal element satisfying ν, i.e.,

$$\min_{\leq}^{\nu}(x) \equiv \nu(x) \wedge \forall y(\nu(y) \rightarrow \neg(y < x)). \qquad (2)$$

When $\leq$ is clear from the context it is omitted as a subscript. Minimal elements in the partial preorder are defined by

$$\min(x) \equiv \forall y(\neg(y < x)).$$

Definition 0.4. *(Translation) The* translation $\tau(P)$ *of a postulate* $\mathcal{P}$ *is the sentence obtained from* $\mathcal{P}$ *by replacing*

1. *every occurrence of* $K(x)$ *with* $\min(x)$
2. *every occurrence of* $\varphi_i(x)$ *and* $\mu_i(x)$ *with their "hat" versions*
3. *every occurrence of* $K * \mu_i$ *with* $\min^{\hat{\mu}_i}(x)$.

Note that Part 2 in the definition is redundant as the definition for φ_i is a special case of the definition for μ_i. The translation is a first-order sentence over the predicate symbols $\leq, A_1, \ldots, A_\ell$.

Example 0.5. *(Translation of postulate (1)) Applying Definition 0.4 we get*

$$[\exists x(\min(x)) \wedge A_1(x)] \rightarrow [\forall x(\min^{A_1}(x) \leftrightarrow (\min(x) \wedge A_1(x)))].$$

Given $K, \varphi_1, \ldots, \varphi_\ell$ and a faithful partial preorder F for K, the $(\varphi_1, \ldots, \varphi_\ell)$-extension of F is determined in the standard way, by interpreting the unary predicate symbols $A_1, \ldots, A_k$ by $A_i(a) = \varphi_i(t(a))$. The following proposition is a direct consequence of the definitions.

Proposition 0.6. *Let* K *be a knowledge base,* $F = (R, t)$ *be a faithful partial preorder for* K *and let* $*_F$ *be the revision operator determined by* F *using minimization. Let* $\varphi_1, \ldots, \varphi_\ell$ *be propositional formulas and* $\mathcal{P}$ *be a postulate. Then* $\mathcal{P}$ *is satisfied by* $*_F$ *for* $\varphi_1, \ldots, \varphi_\ell$ *iff the* $(\varphi_1, \ldots, \varphi_\ell)$*-extension of* F *satisfies* $\tau(P)$.

Characterizability

As we consider partial preorders that are faithful for a knowledge base, we introduce the following property of partial preorders.

Definition 0.7. *(Regular partial preorders) A partial preorder is* regular *if*

1. *every minimal element is smaller than any non-minimal element*
2. *the number of elements is a power of 2.*

An example of a non-regular partial preorder is the 4-element poset with $a < b, c < d$ and no other comparability. Condition 1 is satisfied, for example, if there is a unique minimal element.

Definition 0.8. *($\mathcal{R}$-revision operator) Let* $\mathcal{R}$ *be a family of regular partial preorders. Let* K *be a knowledge base and* $*$ *be a revision operator for* K. *Then* $*$ *is an* $\mathcal{R}$*-revision operator iff there is a faithful partial preorder* $F = (R, t)$ *for* K*, with* $R \in \mathcal{R}$*, representing* $*$ *using minimization.*

In the rest of the paper we restrict the discussion to posets.

Definition 0.9. *A revision operator* $*$ *is* poset-based *if it is of the form* $*_F$ *for some faithful poset* F.

Now we can give the formal definition of characterizability.

Definition 0.10. *(Characterization, characterizability) Let* $\mathcal{R}$ *be a family of regular posets. A finite set of postulates* $\mathcal{P}$ characterizes $\mathcal{R}$*-revision operators if for every knowledge base* K *and every poset-based revision operator* $*$ *for* K *the following holds:* $*$ *satisfies the postulates in* $\mathcal{P}$ *iff* $*$ *is an* $\mathcal{R}$*-revision operator.*

The family of $\mathcal{R}$*-revision operators is* characterizable *if there is a finite set of postulates characterizing* $\mathcal{R}$*-revision operators.*

It may be assumed w.l.o.g. that $\mathcal{P}$ consists of a single postulate.

$\forall MSO_{\min}$ -definability

In this section we develop the concepts and tools for the logical characterization of postulate characterizability.

A *universal monadic second-order ($\forall MSO$) sentence* is of the form

$$\Phi = \forall A_1, \ldots, A_\ell \Psi, \qquad (3)$$

where $A_1, \ldots, A_\ell$ range over unary predicates (or subsets) of the universe, and Ψ is a first-order sentence using the unary predicate symbols $A_1, \ldots, A_\ell$ in addition to the original language (in our case $\leq$ and equality). An existential second-order ($\exists MSO$) sentence is of the form $\Phi = \exists A_1, \ldots, A_\ell \Psi$.

Definition 0.11. *($\forall MSO_{\min}$ sentence) A* $\forall MSO_{\min}$ *sentence is a universal second order sentence where* Ψ *is a first-order sentence using the unary predicate symbols* $A_1, \ldots, A_\ell$ *and the formulas* $\min_{\leq}^{\hat{\mu}}(x)$ *defined in (2), for every Boolean combination* μ *of* $A_1, \ldots, A_\ell$.

Thus $\forall MSO_{\min}$ sentence can use the order relation $\leq$ in a *restricted manner*, only inside a $\min_{\leq}^{\hat{\mu}}$-formula. The definition of $\exists MSO_{\min}$ sentences is analogous.

Definition 0.12. *($\forall MSO_{\min}$ -definability) A family* $\mathcal{R}$ *of regular posets is* $\forall MSO_{\min}$ *-definable if there is a* $\forall MSO_{\min}$ *sentence* Φ *such that for every regular poset* R *it holds that* $R \in \mathcal{R}$ *iff* R *satisfies* Φ.

The logical characterization of postulate characterizability can now be stated.

Theorem 0.13. *Let $\mathcal{R}$ be a family of regular posets. The family of $\mathcal{R}$-revision operators is characterizable iff $\mathcal{R}$ is $\forall MSO_{\min}$ -definable.*

Proof Let $\mathcal{R}$ be a family of regular posets such that $\mathcal{R}$-revision operators are characterized by a postulate $\mathcal{P}$. We claim that $\mathcal{R}$ is defined by the $\forall MSO_{\min}$ sentence

$$\Phi = \forall A_1, \dots A_\ell \ \tau(P).$$

Assume that the regular poset $R = (X, \leq)$ is in $\mathcal{R}$. Let the number of its elements be 2^n. Let $t : X \to \{0,1\}^n$ be an arbitrary bijection between X and the set of truth assignments. We get a faithful poset $F = (R, t)$ for some knowledge base K_n, and thus the corresponding revision operator $*_F$ is an $\mathcal{R}$-revision operator. Therefore $*_F$ satisfies $\mathcal{P}$. Consider arbitrary unary relations $A_1, \dots, A_\ell$ over the elements. Applying Proposition 0.6 to the propositional formulas $\varphi_1, \dots, \varphi_\ell$ corresponding to $A_1, \dots, A_\ell$, it follows that $A_1, \dots, A_\ell$ and the corresponding $\min$-predicates satisfy $\tau(P)$. Thus R satisfies Φ.

Now assume that the regular poset R is not in $\mathcal{R}$. Again, let $t : X \to \{0,1\}^n$ be an arbitrary bijection between X and the set of truth assignments. We get a faithful poset $F = (R, t)$ for some knowledge base K_n. This determines a revision operator $*_F$. We claim that $*_F$ is not an $\mathcal{R}$-revision operator. This follows if we show that F is the only faithful poset determining $*_F$. Assume that for $F' = (R', t')$ with $R' = (X', \leq')$ the revision operator $*_{F'}$ is the same. Then, as revision operators are defined using minimization, for every pair of truth assignments u, v it holds that

$t^{-1}(u) < t^{-1}(v)$ iff $K_n *_F \langle u, v\rangle = \langle u\rangle$ iff $(t')^{-1}(u) < (t')^{-1}(v)$

and

$t^{-1}(u) \sim t^{-1}(v)$ iff $K_n *_F \langle u, v\rangle = \langle u, v\rangle$ iff $(t')^{-1}(u) \sim (t')^{-1}(v)$.

Thus $(t')^{-1} \circ t$ is an isomorphism from R to R'. Hence $*_F$ does not satisfy $\mathcal{P}$. So there are propositional formulas $\varphi_1, \dots, \varphi_\ell$ such that the corresponding instance of $\mathcal{P}$ is false. By Proposition 0.6 the corresponding unary predicates $A_1, \dots, A_\ell$ and their $\min$ predicates falsify $\tau(P)$. Hence R falsifies Φ.

The proof of the other direction is similar. □

Games

The q-round first-order Ehrenfeucht - Fraïssé game over two structures is played by two players, Spoiler and Duplicator. In each round Spoiler picks one of the structures and an element of that structure. Duplicator responds by picking an element in the other structure. After q rounds Duplicator wins if the substructures of picked elements in the two structures are isomorphic. Otherwise Spoiler wins.

In the following definition we use $\exists MSO_{\min}$ instead of $\forall MSO_{\min}$ for convenience; $\exists MSO_{\min}$ -definability of a class is the same as $\forall MSO_{\min}$ -definability of its complement. The game we define is a version of the modification of Ehrenfeucht - Fraïssé games for $\exists MSO$-definability introduced by (Ajtai and Fagin 1990).

Definition 0.14. *($(\mathcal{R}, \ell, q)$-$\exists MSO_{\min}$ game) Given a class $\mathcal{R}$ of posets, the $(\mathcal{R}, \ell, q)$-$\exists MSO_{\min}$ game is played by Spoiler and Duplicator as follows:*

1. *Duplicator picks a poset $R_1 = (X_1, \leq_1)$ in $\mathcal{R}$,*
2. *Spoiler picks ℓ subsets $A_1, \dots, A_\ell$ of X_1,*
3. *Duplicator picks a poset $R_2 = (X_2, \leq_2) \notin \mathcal{R}$ and subsets $B_1, \dots, B_\ell$ of X_2,*
4. *Form the relational structure R_1^* on X_1 with unary relations $A_1, \dots, A_\ell$ and $\min^\nu_{\leq_1}$ for every Boolean combination of the A_is, and the relational structure R_2^* on X_2 with unary relations $B_1, \dots, B_\ell$ and $\min^\nu_{\leq_2}$ for every Boolean combination of the B_is,*
5. *Spoiler and Duplicator play a q-round first-order Ehrenfeucht - Fraïssé game on R_1^* and R_2^*.*

The forgetful property of this game, mentioned in the introduction, is that although R_1^* and R_2^* are defined using the underlying posets, the original order relations $\leq_1$ and $\leq_2$ are "forgotten" and R_1^*, R_2^* are monadic structures.

Theorem 0.15. *A class $\mathcal{R}$ of posets is $\exists MSO_{\min}$ -definable iff there is ℓ and q such that Spoiler wins the $(\mathcal{R}, \ell, q)$ - $\exists MSO_{\min}$ game.*

For the non-characterizability results we use the following corollary. The formulation takes into account that the theorem holds for general posets, but we are only interested in regular posets.

Corollary 0.16. *Let $\mathcal{R}$ be a class of regular posets. Assume that for every ℓ and q, Duplicator has a winning strategy in the $(\mathcal{R}, \ell, q)$ - $\exists MSO_{\min}$ game such that the posets R_2 are also regular. Then $\mathcal{R}$ is not $\exists MSO_{\min}$ -definable.*

Classes of posets

A chain (resp., antichain) is a set of pairwise comparable (resp., incomparable) elements. The *height* (resp., *width*) of a poset is size of a largest chain (resp., antichain). A poset is total (aka linear) if it has width 1. A poset is connected (resp., disconnected) if its comparability graph is connected (resp., disconnected).

As for technical reasons we always deal with regular posets in this paper, we use somewhat modified notions.

The r-height (resp. r-width) of a regular poset is the height (resp., width) of the poset obtained by removing the minimal elements. A regular poset is r-total if the poset obtained by removing the minimal elements is total. A regular poset is r-total (resp., r-connected, r-disconnected) if the poset obtained by removing the minimal elements is total (resp., connected, disconnected).

We denote by $\mathcal{H}_{\leq k}$ the class of regular posets with r-height at most k. The classes $\mathcal{H}_{\geq k}, \mathcal{H}_{<k}, \mathcal{H}_{>k}, \mathcal{H}_{=k}$ are defined similarly. For width we use the notations $\mathcal{W}_{\leq k}, \mathcal{W}_{\geq k}, \mathcal{W}_{<k}, \mathcal{W}_{>k}$. The class of r-total (resp., r-connected, r-disconnected) regular posets is denoted by $\mathcal{T}$ (resp., $\mathcal{C}, \mathcal{D}$).

Characterizable classes

As a first example of characterizability we give a postulate characterizing revision operators obtained from r-total regular posets.

Theorem 0.17. *The class of $\mathcal{T}$-revision operators is characterized by the postulate*

$$(\forall x(\neg(K(x) \wedge \phi(x))) \wedge \exists x\phi(x)) \rightarrow \exists! x((K * \phi)(x)).$$

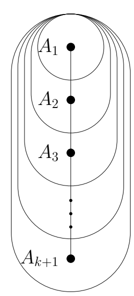
Proof Let $*_R$ be a revision operator generated by an r-total regular poset R. If φ is satisfiable and inconsistent with K then it has a unique minimal satisfying truth assignment and so its revision is a singleton. Conversely, if R is a non-r-total regular poset then it has an incomparable non-minimal pair (a, b). If the revising formula ϕ has the two corresponding truth assignments as its models then the models of the revision by ϕ will consist of these two assignments. □

In the other proofs of this section we use the definability framework. Instead of showing the $\forall MSO_{\min}$ -definability of the corresponding classes of regular posets, we show the $\exists MSO_{\min}$ -definability of the complementary class of regular posets, because it is simpler and more intuitive. We give informal descriptions which can be easily seen to be translatable to formal definitions.

Theorem 0.18. *For every k, the class of $\mathcal{H}_{\leq k}$-revision operators is characterizable.*

Proof We show that $\mathcal{H}_{>k}$ is $\exists MSO_{\min}$ -definable. A regular poset has r-height greater than k iff there exists a chain of $k+1$ non-minimal elements. This is equivalent to the following: there are sets $A_1, \ldots, A_{k+1}$ such that

- the A_is are disjoint from the set of minimal elements,
- there are $k+1$ elements $a_1, \ldots, a_{k+1}$ such that $A_i = \{a_1, \ldots, a_i\}$ for every $i \leq k+1$, and
- $\min A_i = \{a_i\}$. □



Theorem 0.19. *For every k, the class of $\mathcal{H}_{\geq k}$-revision operators is characterizable.*

Proof We show that $\mathcal{H}_{<k}$ is $\exists MSO_{\min}$ -definable. A regular poset has height less than k iff there are sets $A_1, \ldots, A_{k-1}$ such that

- the A_is form a partition of the set of non-minimal elements, and
- each A_i is an antichain.

The property that A_i is an antichain can be expressed by $A_i = \min A_i$. □

Theorem 0.20. *For every k, the class of $\mathcal{W}_{\leq k}$-revision operators is characterizable.*

Proof We show that $\mathcal{W}_{>k}$ is $\exists MSO_{\min}$ -definable. A regular poset has width greater than k iff there is an antichain of size at least $k+1$. This can be expressed similarly to the previous cases. □

Interestingly, the classes $\mathcal{W}_{\leq k}$ are not $\forall MSO_{\min}$ -definable, even though they are first-order. Connectedness and disconnectedness are also not $\forall MSO_{\min}$ -definable. These results are presented in the next section.

On the other hand, connected posets of bounded height turn out to be $\forall MSO_{\min}$ -definable. This is an example of an $\forall MSO_{\min}$ -definable class which is *not* first-order.

Theorem 0.21. *For every k, the class of $\mathcal{H}_{\leq k} \cap \mathcal{C}$-revision operators is characterizable.*

Proof It is sufficient to show that $\mathcal{H}_{=k} \cap \mathcal{D}$ is $\exists MSO_{\min}$ -definable. A regular poset R of r-height k is r-disconnected iff there is a partition of the non-minimal elements into $2k$ antichains $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ such that

- the A_is are all non-empty,
- some B_i is non-empty, and
- every union $A_i \cup B_j$ is also an antichain.

If R has r-height k and is disconnected, then k antichains in an r-height k component, plus at most k antichains for the other components satisfy these conditions. Conversely, assume that R has height k and is connected. Then there is a comparable pair (a, b) such that $a \in A_i$ and $b \in B_j$ for some i, j, contradicting the last condition. □

Non-characterizable classes

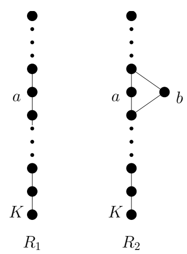
The negative results are based on Theorem 0.13 and Corollary 0.16, by constructing winning strategies for the Duplicator. We first consider the class of $\mathcal{W}_{\geq 2}$-revision operators, i.e., the class of revision operators obtained from regular posets which are *not* r-total. As non r-total regular posets form a simple and natural first-order definable class, the non-characterizability of the corresponding class of revision operators might be considered somewhat surprising.

Theorem 0.22. *The class of $\mathcal{W}_{\geq 2}$-revision operators is not characterizable.*

Proof Given ℓ and q, we have to describe a winning strategy of the Duplicator in the $(\mathcal{R}, \ell, q)$ - $\exists MSO_{\min}$ game for the class of regular posets not in $\mathcal{W}_{\geq 2}$, i.e., for the class $\mathcal{T}$ of r-total regular posets.

The poset $R_1 = (X_1, \leq_1)$ picked in step 1. will be a chain on N elements, for some power of 2 to be determined later. Assume that Spoiler picks ℓ subsets $A_1, \ldots, A_\ell$. For any element $a \in X_1$ we associate a bit-vector with components indicating which subsets A_i a belongs to, and for every Boolean combination ν of the A_is, whether a is minimal among elements belonging to ν. These bit-vectors form a coloring of X_1 with $L = 2^{\ell+2^{2^\ell}}$ colors. As there is at most one element which is minimal in a Boolean combination, there are at least $N - 2^{2^\ell}$ elements which are never minimal. If $N - 2^{2^\ell} > L$ then there are at least two never-minimal elements of the same color. Duplicator then forms R_2 by picking such an element a, splitting it into two incomparable elements a and b, and deleting another element

c from the same color class. (See the figure below for illustration.) The sets B_i are the same as the A_is, with the exception of b replacing c. The monadic structures R_1^* and R_2^* are isomorphic and thus Duplicator can win the first-order game in Step 5. □

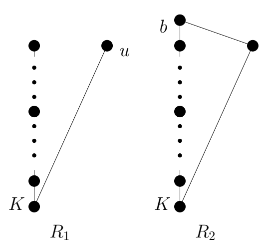


Non-characterizability of the class of $\mathcal{W}_{\geq k}$-revision operators for $k > 2$ follows the same way.

We have seen in Theorem 0.21 that the class of revision operators generated by connected regular posets of bounded height is characterizable. On the other hand, the full class of revision operators generated by connected regular posets turns out to be non-characterizable. This shows another limitation of $\forall MSO_{\min}$ -definability, as connected posets are $\forall MSO$-definable (for every nonempty subset there must be a comparable pair between the subset and its complement).

Theorem 0.23. *The class of $\mathcal{C}$-revision operators is not characterizable.*

Proof The argument is similar to the previous one and it is illustrated by the figure below. Duplicator picks a poset R_1 consisting of a long chain and a single element u. After Spoiler picks the subsets $A_1, \ldots, A_\ell$, Duplicator considers the coloring described above, and finds a color class of never-minimal elements. Duplicator then builds the poset R_2 by picking an element b from this color class and putting it above the chain and u. Then it follows as above that Duplicator can win in Step 5. of the game. □



Finally, we consider the class of revision operators generated by disconnected regular posets.

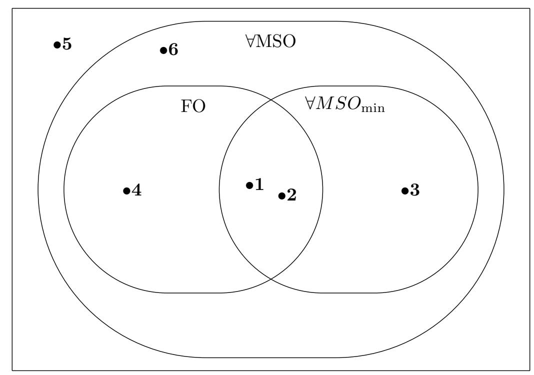
Theorem 0.24. *The class of $\mathcal{D}$-revision operators is not characterizable.*

The theorem follows from this class not even being $\forall MSO$-definable. The proof is based on the standard proof of the undefinability of graph connectivity in $\exists MSO$, and is given in (Turán and Yaggie 2014).

Summary

The results on classes of revision operators are summarized in the figure below. Characterizable classes (denoted by $\forall MSO_{\min}$ in the figure) turn out be a proper subset of universal monadic second-order definable classes, incomparable with first-order definable classes. Some details of the diagram (such as $\mathcal{H}_{\leq k} \cap \mathcal{C}$ not being first-order definable) are omitted due to lack of space. Item 1. contains totality (corresponding to $k = 1$) and Item 4. contains non-totality (corresponding to $k = 2$). Item 2. mentions two different classes.

Characterizable	**Non-characterizable**
1. Width $\leq k$	4. Width $\geq k$
2. Height $\leq k$, height $\geq k$	5. Disconnected
3. Connected and height $\leq k$	6. Connected



In future work we will extend the results of this paper to partial preorders and also to contraction operators. Further questions include characterizability for other types of epistemic information, and for iterated revision.

References

Ajtai, M., and Fagin, R. 1990. Reachability is Harder for Directed than for Undirected Finite Graphs. *J. Symbolic Logic* 55(1):113–150.

Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the Logic of Theory Change: Partial Meet Contraction and Revision Functions. *J. Symbolic Logic* 50(2):510–530.

Ben-Naim, J. 2006. Lack of Finite Characterizations for the Distance-based Revision. In *10th International Conference on Principles of Knowledge Representation and Reasoning (KR'06)*, 239–248.

Blackburn, P.; de Rijke, M.; and Venema, Y. 2001. *Modal Logic*. New York, NY, USA: Cambridge University Press.

Ebbinghaus, H.-D., and Flum, J. 2006. *Finite Model Theory*. Springer Monographs in Mathematics. Berlin: Springer-Verlag, enlarged edition.

Fermé, E., and Hansson, S. O. 2011. AGM 25 years. Twenty-five Years of Research in Belief Change. *J. Philos. Logic* 40(2):295–331.

Grove, A. 1988. Two Modellings for Theory Change. *Journal of Philosophical Logic* 17:157–170.

Hansson, S. O. 1999. *A Textbook of Belief Dynamics*, volume 11 of *Applied Logic Series*. Dordrecht: Kluwer Academic Publishers.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional Knowledge Base Revision and Minimal Change. *Artificial Intelligence* 52(3):263–294.

Lehmann, D. J.; Magidor, M.; and Schlechta, K. 2001. Distance Semantics for Belief Revision. *J. Symb. Log.* 66(1):295–317.

Libkin, L. 2004. *Elements of Finite Model Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin.

Schlechta, K. 2004. *Coherent Systems*. Studies in Logic and Practical Reasoning. Elsevier Science.

Trotter, W. T. 1992. *Combinatorics and Partially Ordered Sets*. Johns Hopkins Series in the Mathematical Sciences. Baltimore, MD: Johns Hopkins University Press.

Turán, G., and Yaggie, J. 2014. Non-characterizability of Belief Revision: an Application of Finite Model Theory. *CoRR* abs/1403.6512.

van Ditmarsch, H.; van der Hoek, W.; and Kooi, B. 2007. *Dynamic Epistemic Logic*. Springer Verlag.