Axiomatizing a Qualitative Principle of Conditional Preservation for Iterated Belief Change

Gabriele Kern-Isberner
Department of Computer Science
TU Dortmund, Germany

Abstract

In their seminal paper, Darwiche and Pearl proposed four axioms for preserving conditional beliefs under iterated belief revision which were recently adapted to iterated belief contraction by Konieczny and Pino Perez. For (semi-)quantitative frameworks like probabilities, Kern-Isberner presented a fully axiomatized principle of conditional preservation for iterated belief change that was shown to cover the axioms both of Darwiche and Pearl, and Konieczny and Pino Perez in the semantic framework of Spohn's ranking function. This paper closes the gap between these works by presenting a purely qualitative principle of conditional preservation for iterated belief change that can be derived from Kern-Isberner's semiquantitative principle and implies all axioms of the mentioned works, showing in particular that iterated belief revision and belief contraction share common methodological grounds which can be adapted by the respective success condition. Moreover, the approach presented in this paper significantly extends the scope of previous works in that it applies to much more general change problems when epistemic states are changed by sets of conditional beliefs.

1 Introduction

AGM theory (Alchourrón, Gärdenfors, and Makinson 1985) laid the foundations of modern belief change theory by providing a frame in terms of postulates for rational belief change, guided by the paradigm that beliefs should not be adopted or given up without justification (*minimal change paradigm*), and that changes should respect logical dependencies. The two major operations of change that AGM theory deals with are revision $*$ and contraction $-$, where revision integrates a new belief A into a prior belief set K, yielding $K * A$ as a result, while contraction is for giving up beliefs A in K, resulting in the posterior belief set $K - A$. The reference point of AGM theory is classical (propositional) logic which ensures a solid logical quality of AGM belief change, but also restricts its scope a lot. Moreover, an AGM change operator is not necessarily a fully binary operator between belief sets and propositions, but usually depends on the prior belief set K, hence iteration of change is difficult and not actually dealt with in AGM theory. Darwiche and Pearl (Darwiche and Pearl 1997) broadened the

AGM framework a lot by implementing its basic postulates for the revision of epistemic states, where revision now is a binary operator taking a prior epistemic state Ψ and a proposition A and returns a revised state $\Psi * A$ in which A is believed. Furthermore, they were the first to observe that following the minimal change paradigm in iterated belief revision may lead to unwanted loss, or establishment of conditional beliefs, and they proposed four further postulates that should ensure that conditional beliefs are neither given up nor adopted in special cases. In this way, they introduced the paradigm of conditional preservation for revising epistemic states as an important supplement to the minimal change paradigm on the propositional level. In (Konieczny and Pino Pérez 2017) build upon Darwiche and Pearl's work to propose advanced postulates for iterated belief contraction, however, without mentioning the idea of conditional preservation.

In (Kern-Isberner 1998), Kern-Isberner axiomatized a principle of conditional preservation for probabilistic belief revision that she transferred to semi-quantitative frameworks later on (Kern-Isberner 2001; 2004), in particular to ordinal conditional functions (OCF) (Spohn 1988; 2012) which have become quite a popular framework for belief change and nonmonotonic reasoning. She showed that her OCF principle of conditional preservation for iterated revision implied Darwiche and Pearl's postulates (Kern-Isberner 1999) and extended them for conditional revision. Moreover, the OCF principle proved also to be very helpful (Kern-Isberner and Krümpelmann 2011; Kern-Isberner and Huvermann 2017) to solve problems related to independence for iterated revision (Jin and Thielscher 2007; Delgrande and Jin 2012). In (Kern-Isberner et al. 2017), the authors proposed an OCF principle of conditional preservation for general belief change and showed that from this, together with mild prerequisites, the postulates of (Konieczny and Pino Pérez 2017) for iterated contraction can be derived. Therefore, the axiomatized principle of conditional preservation for (semi-)quantitative frameworks presented in (Kern-Isberner 2004) could be considered as a very fundamental and strong universal principle for (iterated, advanced) belief change if it were not based crucially on numbers and arithmetic operations which might be principally responsible for the impact of the principle. Moreover, the relevance of this principle for purely qualitative frameworks,

i.e., where epistemic states are simply equipped with total preorders, seems to be questionable.

In this paper, we now present a purely qualitative version of conditional preservation for iterated belief change which can easily be derived from that of (Kern-Isberner et al. 2017) since it is much weaker, but still strong enough to imply all postulates of (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017), when combined with the respective success axioms for revision and contraction and with stability axioms that help to ensure minimal change for iterated revision. In spite of being purely qualitative, our principle is nevertheless applicable in very advanced belief change scenarios where epistemic states have to be revised by sets of conditional beliefs, thus going far beyond the scenarios considered in (Darwiche and Pearl 1997; Konieczny and Pino Pérez 2017). Indeed, exploiting consequently the concept of conditionals as basic entities of belief for belief change problems not only helps overcoming the narrow bounds of propositional logic that have limited AGM theory but proves to be a unifying paradigm solving different belief change tasks on the base of a single, powerful principle which can be supplemented by other axioms to meet specific demands.

The main contributions of this paper are as follows:

- We present axiomatic guidelines for iterated change of epistemic states (equipped with total preorders) by sets of conditional beliefs, thus covering also the cases where an epistemic state is changed by a single conditional, or by a single proposition.

- All axioms for iterated revision by (Darwiche and Pearl 1997) as well as all axioms for iterated contraction by (Konieczny and Pino Pérez 2017) can be derived from just one principle, together with respective success axioms and a basic principle of minimal change.

- Our principle of conditional preservation can be considered as a purely qualitative version of the corresponding principle for ordinal conditional functions published in (Kern-Isberner et al. 2017).

This paper is organized as follows: In the next section 2, we summarize formal preliminaries regarding the used propositional logic, conditionals, ordinal conditional functions, and epistemic states in general. Section 3 recalls mainly the results from (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017) which are most relevant for this paper. In section 4, the novel qualitative principle of conditional preservation for iterated belief change is developed and is related to the OCF principle from (Kern-Isberner et al. 2017). Section 5 focuses on the case that epistemic states are changed by just one conditional, resp. just one propositional belief, and hence relates our approach to the ones of (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017). In Section 6, we summarize the results of this paper and point out future work.

2 Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language, with atoms represented by $a, b, c, \ldots$, and with formulas represented by $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connector, writing AB instead of $A \wedge B$, and overlining propositions will indicate negation, i.e. $\overline{A}$ means $\neg A$. If A is a formula, then $\dot{A}$ is any of A or $\overline{A}$. Let Ω denote the set of possible worlds over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$. The *models* of A are given by $Mod(A) = \{\omega \mid \omega \models A\}$. If $A, B \in \mathcal{L}$ are formulas, then B is a consequence of A, in symbols: $A \models B$, iff $Mod(A) \subseteq Mod(B)$, and A and B are *semantically equivalent*, $A \equiv B$, iff $Mod(A) = Mod(B)$. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. Given a set of possible worlds $\Omega' \subseteq \Omega$, $\mathcal{T}(\Omega') = \{A \in \mathcal{L} \mid \omega \models A \text{ for all } \omega \in \Omega'\}$ denotes the set of formulas which are true in all elements of Ω'.

By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of conditionals over $\mathcal{L}$. $(B|A)$ formalizes "*if A then usually B*" and establishes a plausible connection between the *antecedent* A and the *consequent* B. As to the semantics of conditionals, we follow basically the approach of de Finetti (DeFinetti 1974) who considered conditionals as *generalized indicator functions*:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models AB \\ 0 & : \quad \omega \models A\overline{B} \\ u & : \quad \omega \models \overline{A} \end{cases} \quad (1)$$

where u stands for *unknown* or *indeterminate*. Hence, conditionals are three-valued logical entities and thus extend the binary setting of classical logics substantially. Two conditionals $(B|A), (D|C)$ are *equivalent* according to (DeFinetti 1974) if they result in the same indicator function, i.e., iff $AB \equiv CD$ and $A\overline{B} \equiv C\overline{D}$. In this sense, conditionals with tautological antecedents are equivalent to propositional statements, so we may identify the conditional $(A|\top)$ with the proposition A.

Ordinal conditional functions (OCFs), (also called *ranking functions*) $\kappa : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, were introduced (in a more general form) first by (Spohn 1988). They express degrees of (im)plausibility of possible worlds under the convention that lower degrees mean more plausible worlds. So, most plausible worlds have rank 0, and for consistency reasons, there must be at least one such world. With $[\![\kappa]\!] = \{\omega \mid \kappa(\omega) = 0\}$, we denote the set of most plausible worlds of κ. Also propositional formulas A are assigned degrees of disbelief by defining $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. A conditional $(B|A)$ is *accepted* in the epistemic state represented by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$.

In more general settings, *epistemic states* Ψ will be represented by a total preorder $\preceq_\Psi$ on Ω which is most suitable in the context of belief revision (cf. Section 3). As usual, $\omega_1 \prec_\Psi \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$ and not $\omega_2 \preceq_\Psi \omega$, and $\omega_1 \approx_\Psi \omega_2$ iff both $\omega_1 \preceq_\Psi \omega_2$ and $\omega_2 \preceq_\Psi \omega_1$. In a natural way $\preceq_\Psi$ can be lifted to a total preorder on the set of propositions via $A \preceq_\Psi B$ iff for all $\omega_2 \in Mod(B)$, there is $\omega_1 \in Mod(A)$

such that $\omega_1 \preceq_\Psi \omega_2$; for total preorders, this is equivalent to saying that there is an $\omega_1 \in Mod(A)$ such that $\omega_1 \preceq_\Psi \omega_2$ for all $\omega_2 \in Mod(B)$. $A \prec_\Psi B$ and $A \approx_\Psi B$ are defined in the same way as above.

If $\Omega' \subseteq \Omega$, then $\min_{\preceq_\Psi}(\Omega') = \{\omega_1 \in \Omega' \mid \omega_1 \preceq_\Psi \omega_2$ for all $\omega_2 \in \Omega'\}$ denotes the set of minimal models in Ω' with respect to $\preceq_\Psi$. If $\Omega' = \Omega$, then we simply write $\min(\Psi)$ instead of $\min_{\preceq_\Psi}(\Omega)$. If $A \in \mathcal{L}$, then $\min_{\preceq_\Psi}(A) = \min_{\preceq_\Psi}(Mod(A))$. The minimal models of an epistemic state induce its associated belief set: $Bel(\Psi) = \mathcal{T}(\min(\Psi))$, i.e., the agent believes exactly the propositions that are valid in all most plausible worlds.

Similarly to what holds for OCFs, a conditional $(B|A)$ is *accepted* by the epistemic state Ψ, $\Psi \models (B|A)$, iff $AB \prec_\Psi A\overline{B}$. This means that, by definition, Ψ accepts all conditionals $(B|A)$ such that $A\overline{B}$ is contradictory while AB is not, and does not accept any conditional $(B|A)$ where AB is contradictory. One might extend this definition to also cover limiting cases where the antecedent of a conditional is contradictory by defining that any such conditional be accepted, but this would be an artefact reminding of material implication which is neither justified in our semantic framework of conditionals, nor is it necessary. Please note that our conditional approach sketched in the beginning of section 4 also works for conditionals $(B|A)$ where AB or $A\overline{B}$ are contradictory. However, such conditionals may lead easily to trivial belief change tasks because revision resp. contraction by such conditionals is either vacuous, or not possible at all. Therefore, the focus of this paper is on "most normal" conditionals $(B|A)$ where both AB and $A\overline{B}$ are non-contradictory because our aim is to show how relationships between verifying (i.e., $\omega \models AB$) and falsifying worlds (i.e., $\omega \models A\overline{B}$) are influenced by belief change via the principle of conditional preservation.

Note that also OCFs κ induce total preorders on Ω via $\omega_1 \preceq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$, so everything we state on general epistemic states will apply to OCFs, but OCFs allow for more expressive statements because of their usage of natural numbers and the corresponding arithmetics. For an OCF κ, we have accordingly $Bel(\kappa) = \mathcal{T}([\![\kappa]\!])$; thus, a proposition A is believed under κ if $\kappa(\overline{A}) > 0$ (which implies particularly $\kappa(A) = 0$).

3 Basics on Belief Change and Related Work

AGM revision of an epistemic state Ψ (in the sense of (Darwiche and Pearl 1997) can be ensured by assuming that a so-called faithful ranking underlies Ψ such that the revised beliefs can be computed from the minimal models according to the ranking. Here, a faithful ranking is a total preorder $\preceq$ on the possible worlds that is assigned to Ψ in such a way that the minimal models of $\preceq$ are precisely the models of the belief set $K = Bel(\Psi)$ associated with Ψ, containing the most plausible beliefs of Ψ.

Proposition 1 ((Darwiche and Pearl 1997)) *A revision operator $*$ that assigns a posterior epistemic state $\Psi * A$ to a prior state Ψ and a proposition A is an AGM revision operator for epistemic states extending propositional revision on $K = Bel(\Psi)$ iff there exists a faithful preorder*

$\preceq$ *such that for every proposition C it holds that:*

$$K * C = Bel(\Psi * C) = \mathcal{T}(\min(\preceq, C))$$

This proposition allows us to study AGM-style revisions by focussing on total preorders assigned to epistemic states Ψ, henceforth denoted by $\preceq_\Psi$. As pointed out by Darwiche and Pearl, and others, some of the revisions characterised by Proposition 1 lead to unintuitive results in the case of iterated revision. An iterative revision operator $*$ should fulfill further postulates, especially those that ensure that the ordering of specific worlds is kept:

Proposition 2 ((Darwiche and Pearl 1997)) *Let $*$ be an AGM revision operator for epistemic states Ψ with corresponding faithful preorder $\preceq_\Psi$. Then $*$ is an iterative revision operator in the sense of (Darwiche and Pearl 1997) iff for every proposition C it holds that:*

(DP1) *If $\omega_1, \omega_2 \models C$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi*C} \omega_2$.*

(DP2) *If $\omega_1, \omega_2 \models \overline{C}$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi*C} \omega_2$.*

(DP3) *If $\omega_1 \models C$ and $\omega_2 \models \overline{C}$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi*C} \omega_2$.*

(DP4) *If $\omega_1 \models C$ and $\omega_2 \models \overline{C}$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi*C} \omega_2$.*

This approach by Darwiche and Pearl has been widely accepted, and is the basis for many results on iterated belief revision.

Some authors have transferred the semantic postulates of (Darwiche and Pearl 1997) to the framework of iterated contraction (Chopra et al. 2008; Ramachandran, Nayak, and Orgun 2012; Konieczny and Pino Pérez 2017)[1], we recall the postulates of (Konieczny and Pino Pérez 2017) here:

Proposition 3 ((Konieczny and Pino Pérez 2017)) *Let $-$ be an AGM contraction operator for epistemic states Ψ with corresponding faithful preorder $\preceq_\Psi$. Then $-$ is an iterative contraction operator in the sense of (Konieczny and Pino Pérez 2017) iff for every proposition C it holds that:*

(KPP1) *If $\omega_1, \omega_2 \models C$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi-C} \omega_2$.*

(KPP2) *If $\omega_1, \omega_2 \models \overline{C}$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi-C} \omega_2$.*

(KPP3) *If $\omega_1 \models \overline{C}$ and $\omega_2 \models C$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi-C} \omega_2$.*

(KPP4) *If $\omega_1 \models \overline{C}$ and $\omega_2 \models C$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi-C} \omega_2$.*

For further explanations of the postulates of (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017), for their syntactic counterparts and illustrations, please see the original papers.

The principle of conditional preservation for multiple iterated change operators $\circ$, OCFs, and sets $\mathcal{R}$ of conditionals of (Kern-Isberner et al. 2017) is also highly relevant, but since it shares common theoretical grounds with the novel qualitative principle of conditional preservation to be developed in this paper, we postpone recalling it to the next section.

---

4 A Qualitative Principle of Conditional Preservation

In this section, we will study the following iterated change problem:

> Given an epistemic state Ψ, building upon a logical language $\mathcal{L}$ and equipped with a total preorder $\preceq_\Psi$ on its possible worlds Ω, a set $\mathcal{R} = \{(B_1|A_1),\ldots,(B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ of conditional beliefs, and a change operator $\circ$, how should Ψ be changed in a rational way to yield a posterior state $\Psi^\circ = \Psi \circ \mathcal{R}$ (also equipped with a total preorder) such that the change complies with the characteristics of $\circ$ and such that conditional beliefs in Ψ and $\mathcal{R}$ are treated adequately? In particular, conditional beliefs should be preserved if there is no reason to give them up.

In order to not trivialize the change task, we assume $\mathcal{R}$ to be consistent in the following, i.e., there should be at least one total preorder on Ω that accepts all conditionals in $\mathcal{R}$.

The idea of rational change will be ensured by starting from an AGM framework resp. the extensions of AGM provided by (Darwiche and Pearl 1997), i.e., by obeying Proposition 1. The characteristics of $\circ$ are given by the respective success conditions in the first place, i.e., if $\circ$ is a revision operator, then we expect that $\Psi^\circ \models \mathcal{R}$, while for a contraction, we expect $\Psi^\circ \not\models \mathcal{R}$ in the sense, that $\Psi^\circ \not\models (B|A)$ for any conditional in $\mathcal{R}$; however, one may consider further characteristics here. The most problematic issue to be addressed is to devise a strategy how to preserve conditional beliefs. Of course, there are simple cases, for example, if $\Psi \models (B|A)$, $(\overline{B}|A) \in \mathcal{R}$ and $\circ = *$ is a revision operator, then we expect that $(B|A)$ is overridden by the conditional beliefs in $\mathcal{R}$. But there are far more complicated cases – what if $\Psi \models (B|A)$ and $\mathcal{R}$ contains conditionals such as $(A|B), (C|AB)$, or $(\overline{B}|AC)$? Then the question of which conditional beliefs in Ψ are affected by the new conditional beliefs in $\mathcal{R}$ and which are not, is not at all trivial. And indeed, the problem is even worse: Not only the beliefs in Ψ vs. those in $\mathcal{R}$ interact, also the conditional beliefs within Ψ resp. $\mathcal{R}$ interact, they may support one another, like $(B|A)$ and $(A|B)$ (cf. (Eichhorn, Kern-Isberner, and Ragni 2018)), or state exceptions to other conditionals, like $(B|A)$ and $(\overline{B}|AC)$.

The theory of conditional structures (Kern-Isberner 2001; 2004) provides a suitable basis to make these interactions transparent. Since we base also our qualitative principle of conditional preservation on this algebraic framework, we recall its basic facts which are relevant for this paper. Conditional structures are kind of labels that are assigned to possible worlds in order to reveal clearly how the world evaluates all conditionals in $\mathcal{R}$. Basically, we follow the approach (1), but since we deal with multiple conditionals at the same time, we have to be able to tell their influences on the possible world apart. We solve this problem by assigning different algebraic symbols to different conditionals, thus treating each conditional as an *independent piece of information* (see also (Kern-Isberner and Huvermann 2017)).

More formally, let $\mathcal{R} = \{(B_1|A_1),\ldots,(B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ be a finite set of conditionals, and let $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$ be distinct algebraic symbols that are used as generators of a (free abelian[2]) group (Fine and Rosenberger 1999) $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$. In short, this group structure provides us with a multiplication (written as juxtaposition) and with a neutral element 1 that symbolizes non-applicability. Furthermore, the property of free abelian makes a parallel handling of the conditionals (without any order of application assumed) possible, as well as independence between different conditionals (by forbidding cancellations between different symbols). In extending the basic idea of (1) to the case of multiple conditionals, for each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i = \sigma_{(B_i|A_i)} : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) := \begin{cases} \mathbf{a}_i^+ & \text{if} \quad \omega \models A_i B_i & \text{(verification)} \\ \mathbf{a}_i^- & \text{if} \quad \omega \models A_i \overline{B_i} & \text{(falsification)} \\ 1 & \text{if} \quad \omega \models \overline{A_i} & \text{(non-applicability)} \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to the possible world ω. The function $\sigma_\mathcal{R} : \Omega \to \mathcal{F}_\mathcal{R}$ given by

$$\sigma_\mathcal{R}(\omega) := \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \mathbf{a}_i^-$$

describes the all-over effect of $\mathcal{R}$ on ω. Furthermore, $\sigma_\mathcal{R}(\omega)$ is called the *conditional structure of ω with respect to $\mathcal{R}$*. Since $\mathcal{F}_\mathcal{R}$ is a free (abelian) group, the conditional structures of worlds are uniquely determined by their σ_i-components and hence by their logical relation to each conditional: For any two worlds ω_1, ω_2, we have

$$\sigma_\mathcal{R}(\omega_1) = \sigma_\mathcal{R}(\omega_2) \quad \text{iff} \quad \sigma_i(\omega_1) = \sigma_i(\omega_2) \tag{2}$$
$$\text{for all } i, 1 \leqslant i \leqslant n.$$

The following simple example illustrates the notion of conditional structures and shows how to calculate in this framework:

Example 1 *Let $\mathcal{R} = \{(c|a), (c|b)\}$, where a, b, c are atoms, and let $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \mathbf{a}_2^+, \mathbf{a}_2^- \rangle$. We associate $\mathbf{a}_1^\pm$ with the first conditional, $(c|a)$, and $\mathbf{a}_2^\pm$ with the second one, $(c|b)$. For instance, the world abc verifies both conditionals, so we have $\sigma_\mathcal{R}(abc) = \mathbf{a}_1^+ \mathbf{a}_2^+$. The following table shows the values of the function $\sigma_\mathcal{R}$ on arbitrary worlds $\omega \in \Omega$:*

ω	$\sigma_\mathcal{R}(\omega)$	ω	$\sigma_\mathcal{R}(\omega)$	ω	$\sigma_\mathcal{R}(\omega)$	ω	$\sigma_\mathcal{R}(\omega)$
abc	$\mathbf{a}_1^+ \mathbf{a}_2^+$	$\overline{a}bc$	$\mathbf{a}_2^+$	$ab\overline{c}$	$\mathbf{a}_1^- \mathbf{a}_2^-$	$\overline{a}b\overline{c}$	$\mathbf{a}_2^-$
$a\overline{b}c$	$\mathbf{a}_1^+$	$\overline{a}\overline{b}c$	1	$a\overline{b}\overline{c}$	$\mathbf{a}_1^-$	$\overline{a}\overline{b}\overline{c}$	1

We find that $\sigma_\mathcal{R}(abc) \cdot \sigma_\mathcal{R}(\overline{a}\overline{b}c) = \mathbf{a}_1^+ \mathbf{a}_2^+ \cdot 1 = (\mathbf{a}_1^+) \cdot (\mathbf{a}_2^+) = \sigma_\mathcal{R}(a\overline{b}c) \cdot \sigma_\mathcal{R}(\overline{a}bc)$, which may be interpreted by saying that the sets of worlds $\{abc, \overline{a}\overline{b}c\}$ and $\{a\overline{b}c, \overline{a}bc\}$ show identical conditional effects – they are balanced with respect to the effects of the conditionals in $\mathcal{R}$. Although $abc, \overline{a}\overline{b}c, a\overline{b}c, \overline{a}bc$ all have different conditional structures, the relationships between them with respect to $\mathcal{R}$ are clearly revealed.

---

[2]Free abelian groups have no relations except for those induced by commutativity.

The last example shows that not only the conditional structures of single worlds are informative but that also that (multi)sets[3] of worlds can form sort of coalitions to evaluate the conditionals in $\mathcal{R}$. So, we generalize the definition of conditional structures to multisets, making use of the multiplicative structure within $\mathcal{F}_\mathcal{R}$:

Definition 1 *Let* $[\Omega_1] = \{\omega_1, \ldots, \omega_m\}$ *be a multiset. Then*

$$\sigma_\mathcal{R}([\Omega_1]) = \sigma_\mathcal{R}(\omega_1) \cdots \sigma_\mathcal{R}(\omega_m) \qquad (3)$$

is the conditional structure of $[\Omega]$. *If* $[\Omega_1]$ *does not contain multiple copies of elements and hence is simply a set, we write* $\sigma_\mathcal{R}(\Omega_1)$.

Example 2 *In Example 1, for the sets* $\Omega_1 = \{abc, \overline{a}\overline{b}c\}$ *and* $\Omega_2 = \{a\overline{b}c, \overline{a}bc\}$, *we have* $\sigma_\mathcal{R}(\Omega_1) = \sigma_\mathcal{R}(\Omega_2) = \mathbf{a}_1^+ \mathbf{a}_2^+$.

Lemma 1 *Let* $[\Omega_1] = \{\omega_1, \ldots, \omega_m\}$ *be a multiset of worlds. Then*

$$\sigma_\mathcal{R}([\Omega_1]) = \prod_{1 \leqslant i \leqslant n} (\mathbf{a}_i^+)^{|\{j : \sigma_i(\omega_j) = \mathbf{a}_i^+\}|}$$
$$\cdot \prod_{1 \leqslant i \leqslant n} (\mathbf{a}_i^-)^{|\{j : \sigma_i(\omega_j) = \mathbf{a}_i^-\}|},$$

where $| \cdot |$ *denotes the cardinality of a multiset, i.e., multiple occurrences of worlds are counted.*

Let $[\Omega_2] = \{\omega_1', \ldots, \omega_m'\}$ *also be a multiset of worlds with the same cardinality as* $[\Omega_1]$. *Then* $\sigma_\mathcal{R}([\Omega_1]) = \sigma_\mathcal{R}([\Omega_2])$ *iff for all* $i, 1 \leqslant i \leqslant n$, $|\{j : \sigma_i(\omega_j) = \mathbf{a}_i^+\}| = |\{j : \sigma_i(\omega_j') = \mathbf{a}_i^+\}|$ *and* $|\{j : \sigma_i(\omega_j) = \mathbf{a}_i^-\}| = |\{j : \sigma_i(\omega_j') = \mathbf{a}_i^-\}|$, *i.e., iff for each conditional* $(B_i|A_i)$ *in* $\mathcal{R}$, Ω_1 *and* Ω_2 *show the same number of verifications resp. falsifications.*

Proof. Since $\mathcal{F}_\mathcal{R}$ is commutative, the generators $\mathbf{a}_i^+$ and $\mathbf{a}_i^-$ can be rearranged, and multiple occurrences are expressed by suitable exponents. This proves the first statement. Moreover, they are also free generators, which means that there are no cancellations between different $\mathbf{a}_i^+$, $\mathbf{a}_i^-$, hence two elements of $\mathcal{F}_\mathcal{R}$ can be the same only if their exponents of each generator coincide. This shows the second statement. □

With these notations, a principle of conditional preservation for multiple iterated change operators ∘, OCFs, and sets $\mathcal{R}$ of conditionals was presented in (Kern-Isberner et al. 2017):

(**PCP**$_\circ^{ocf}$) Let ∘ be a change operation for OCFs κ and conditional belief sets $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$. If two multisets of possible worlds $[\Omega_1] = \{\omega_1, \ldots, \omega_m\}$ and $[\Omega_2] = \{\omega_1', \ldots, \omega_m'\}$ with the same cardinality fulfill $\sigma_\mathcal{R}([\Omega_1]) = \sigma_\mathcal{R}([\Omega_2])$ then prior κ and posterior $\kappa^\circ = \kappa \circ \mathcal{R}$ are related by

$$(\kappa(\omega_1) + \ldots + \kappa(\omega_m)) - (\kappa(\omega_1') + \ldots + \kappa(\omega_m'))$$
$$= (\kappa^\circ(\omega_1) + \ldots + \kappa^\circ(\omega_m)) - (\kappa^\circ(\omega_1') + \ldots + \kappa^\circ(\omega_m')) \qquad (4)$$

Rearranging (4) shows that (PCP$_\circ^{ocf}$) guarantees that the overall amount of change within the two multisets measured by the (iterated) differences between prior and posterior κ-value for each world is the same if they behave the same with respect to the new conditional beliefs, i.e., if

$$(\kappa(\omega_1) - \kappa^\circ(\omega_1)) + \ldots + (\kappa(\omega_m) - \kappa^\circ(\omega_m))$$
$$= (\kappa(\omega_1') - \kappa^\circ(\omega_1')) + \ldots + (\kappa(\omega_m')) - \kappa^\circ(\omega_m')) \qquad (5)$$

For relations of this property to independence properties of iterated belief revision, please see (Kern-Isberner and Huvermann 2017).

Of course, (PCP$_\circ^{ocf}$) requires the ranking framework to provide basic arithmetic features. The following characterisation of the change of ranking functions under the principle of conditional preservation was also presented in (Kern-Isberner et al. 2017):

Theorem 1 ((Kern-Isberner et al. 2017)) *Let* $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ *be a finite set of conditionals, and let* $\kappa \circ \mathcal{R} = \kappa^\circ$ *be a belief change of* κ *by* $\mathcal{R}$. *Then this change satisfies* (PCP$_\circ^{ocf}$) *iff there are rational[4] numbers* $\kappa_0, \gamma_i^+, \gamma_i^-, 1 \leqslant i \leqslant n$ *such that*

$$\kappa^\circ(\omega) = \kappa_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \gamma_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \gamma_i^-. \qquad (6)$$

Iterated belief change operators of the form (6) *are called* c-change operators.

γ_i^+ and γ_i^- are constants associated with each conditional affecting verifying and falsifying worlds in a uniform way, and κ_0 is a normalizing constant ensuring that κ° is an OCF, i.e. there is at least one world ω such that $\kappa^\circ(\omega) = 0$.

Transferring these ideas to the purely qualitative case where we consider a general epistemic state Ψ equipped with a total preorder $\preceq_\Psi$, we are no longer able to make use of any arithmetic operation like addition or subtraction (which were essential for formalizing (4)) but may only compare worlds to one another. For this, rearranging the differences in (4) equivalently yields an equation that is much more helpful for the purely qualitative case:

$$(\kappa(\omega_1) - \kappa(\omega_1')) + \ldots + (\kappa(\omega_m) - \kappa(\omega_m')) =$$
$$= (\kappa^\circ(\omega_1) - \kappa^\circ(\omega_1')) + \ldots + (\kappa^\circ(\omega_m) - \kappa^\circ(\omega_m')). \qquad (7)$$

Equation (7) says that (aggregated) prior and posterior differences between worlds from $[\Omega_1]$ and $[\Omega_2]$ are perfectly balanced, given that both multisets satisfy the necessary prerequisites. Without the addition operation, we can hardly express a perfect balance except for trivial cases but we can control tendencies. Observing that $\kappa(\omega) - \kappa(\omega') \leqslant 0$ iff $\omega \preceq_\kappa \omega'$, we obtain an important consequence of (PCP$_\circ^{ocf}$) which makes only use of comparisons:

If there is i s.t. $\omega_i \prec_\kappa \omega_i'$, and for all $j \neq i$, $\omega_j \preceq_\kappa \omega_j'$, then there is j s.t. $\omega_j \prec_{\kappa^\circ} \omega_j'$. (8)

An analogous statement can be derived from (7) with $\preceq_\kappa$ and $\preceq_{\kappa^\circ}$ interchanged. Notably, we can still maintain a very

---

[3]I.e., the worlds need not all be distinct.

[4]Note that indeed, $\kappa_0, \gamma_i^+, \gamma_i^-$ can be rational, but κ° has to satisfy the requirements for OCF, in particular, all $\kappa^\circ(\omega)$ must be non-negative integers.

general view on belief change not being forced to specify whether the general change operator $\circ$ is to perform revision or contraction in advance. Based on these intuitions, we can now formalize a *qualitative principle of conditional preservation for iterated belief change* by slightly rephasing (8) for general epistemic states Ψ:

(**QPCP$_\circ$**) Let $\circ$ be a change operation for epistemic states Ψ and conditional knowledge bases $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$. If for two multisets of possible worlds $[\Omega_1] = \{\omega_1, \ldots, \omega_m\}$ and $[\Omega_2] = \{\omega'_1, \ldots, \omega'_m\}$ with the same cardinality, we have $\sigma_{\mathcal{R}}([\Omega_1]) = \sigma_{\mathcal{R}}([\Omega_2])$ then prior Ψ and posterior $\Psi^\circ = \Psi \circ \mathcal{R}$ fulfill the following two conditions:

(1) If there is $i, 1 \leqslant i \leqslant m$, such that $\omega_i \prec_\Psi \omega'_i$ holds, then there is $j, 1 \leqslant j \leqslant m$, such that $\omega_j \prec_{\Psi\circ} \omega'_j$ holds, or there is $j \neq i$ such that $\omega'_j \prec_\Psi \omega_j$.

(2) If there is $i, 1 \leqslant i \leqslant m$, such that $\omega_i \prec_{\Psi\circ} \omega'_i$ holds, then there is $j, 1 \leqslant j \leqslant m$, such that $\omega_j \prec_\Psi \omega'_j$ holds, or there is $j \neq i$ such that $\omega'_j \prec_{\Psi\circ} \omega_j$.

Expressed in simple words, (QPCP$_\circ$) says that if all pairs (ω_i, ω'_i) are oriented in the same way with respect to $\preceq$ in the prior (resp., posterior) epistemic state, and there is at least one pair that makes a difference, then at least one such pair of worlds with the same orientation can be found in the posterior (resp., prior) state that makes a difference, too. Since no specific order of elements is assumed in $[\Omega_1]$ and $[\Omega_2]$, any arrangement of pairs from both multisets is taken into account. So, in the qualitative case, it is even clearer that we need to consider multisets because we actually consider pairs (ω_i, ω'_i) resp. relations $\omega_i \preceq \omega'_i$ which are compared to one another, and one and the same world can be compared to different worlds, i.e., can occur several times in such pairs. This happens quite naturally when dealing with the transivity of $\preceq_\Psi$. For instance, consider a case where we have $\mathcal{R} = \{A\}$, and $\omega_1 \prec_\Psi \omega_2 \prec_\Psi \omega_3 \prec_\Psi \omega_4$ such that $\omega_1, \omega_4 \models A$, while $\omega_2, \omega_3 \models \neg A$. Then we might want to consider the implications for $\prec_{\Psi\circ}$ of the comparisons $\omega_1 \prec_\Psi \omega_2, \omega_1 \prec_\Psi \omega_3, \omega_2 \prec_\Psi \omega_4, \omega_3 \prec_\Psi \omega_4$ which are implied by transitivity. The multisets $[\Omega_1] = \{\omega_1, \omega_1, \omega_2, \omega_3\}$ and $[\Omega_2] = \{\omega_2, \omega_3, \omega_4, \omega_4\}$ have the same conditional structure with respect to $\mathcal{R}$ and thus can be used to apply (QPCP$_\circ$). Each of the multisets mentions one world twice, and this is needed to capture the implications of transitivity. However, since it is mandatory to consider all arrangements of worlds in pairs with one world from $[\Omega_1]$ and one world from $[\Omega_2]$, the order in the multisets does not matter, and so we stick to our set notation. Note that this principle applies to most general change scenarios, since it covers not only iterated change but also multiple and conditional change and therefore goes far beyond the scope of previous works but indeed, makes only use of comparisons between worlds.

It is straightforward to check that (QPCP$_\circ$) is a consequence of (PCP$_\circ^{ocf}$):

Proposition 4 *(PCP$_\circ^{ocf}$) implies (QPCP$_\circ$), i.e., every belief change operator for OCFs and conditional belief sets that satisfies (PCP$_\circ^{ocf}$) also satisfies (QPCP$_\circ$).*

Proof. This is indeed clear thanks to the equivalence of (4) and (7), and the fact that (8) is an easy consequence of (7). (QPCP$_\circ$)(1) is then an equivalent rephrasing of (8). In the same way, (QPCP$_\circ$)(2) can be derived from (7). $\square$

Remark. In the given form, (QPCP$_\circ$) can also be used as a guideline for changing *partial* preorders $\prec_\Psi$ to *partial* preorders $\prec_{\Psi\circ}$ by sets of conditionals.

Contrapositions of the axioms (1) and (2) of (QPCP$_\circ$) yield axioms for $\preceq_\Psi$ and $\preceq_{\Psi\circ}$:

Proposition 5 *A change operator $\circ$ for epistemic states Ψ and conditional knowledge bases $\mathcal{R}$ satisfies (QPCP$_\circ$) iff for any two multisets of possible worlds $[\Omega_1] = \{\omega_1, \ldots, \omega_m\}$ and $[\Omega_2] = \{\omega'_1, \ldots, \omega'_m\}$ with the same cardinality and the same conditional structure $\sigma_{\mathcal{R}}([\Omega_1]) = \sigma_{\mathcal{R}}([\Omega_2])$, prior Ψ and posterior $\Psi^\circ = \Psi \circ \mathcal{R}$ fulfill the following two conditions:*

(**1'**) *If there is $i, 1 \leqslant i \leqslant m$, such that $\omega_i \preceq_\Psi \omega'_i$ holds, then there is $j, 1 \leqslant j \leqslant m$, such that $\omega_j \prec_{\Psi\circ} \omega'_j$ holds, or for all $i, 1 \leqslant i \leqslant m$, we have $\omega_i \preceq_{\Psi\circ} \omega'_i$, or there is $j \neq i$ such that $\omega'_j \prec_\Psi \omega_j$.*

(**2'**) *If there is $i, 1 \leqslant i \leqslant m$, such that $\omega_i \preceq_{\Psi\circ} \omega'_i$ holds, then there is $j, 1 \leqslant j \leqslant m$, such that $\omega_j \prec_\Psi \omega'_j$ holds, or for all $i, 1 \leqslant i \leqslant m$, we have $\omega_i \preceq_\Psi \omega'_i$, or there is $j \neq i$ such that $\omega'_j \prec_{\Psi\circ} \omega_j$.*

Proof. (1') is the contrapositive form of (QPCP$_\circ$)(2), and (2') is the contrapositive form of (QPCP$_\circ$)(1). $\square$

We refer to conditions (1') resp. (2') of Proposition 5 as (QPCP$_\circ$)(1') resp. (QPCP$_\circ$)(2') in the following.

In the next section, we focus on the case where an epistemic state is changed by a single conditional, and we show connections to the works (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017).

5 Changing Epistemic States By a Single Conditional

In the following, we derive from (QPCP$_\circ$) first general results of changing epistemic states by sets $\mathcal{R} = \{(B|A)\}$ consisting of a single conditional, and then show consequences for the specific case of iterated revision and contraction. In this case, $\mathcal{F}_{\mathcal{R}}$ is the free abelian group that is generated by $\mathbf{a}_1^+, \mathbf{a}_1^-$. Since propositions A can be identified with the conditional $(A|\top)$, these results are immediately relevant also for the (usually considered) case where epistemic states are changed by single propositions.

Proposition 6 *Let $\circ$ be a change operator for epistemic states Ψ equipped with a total preorder $\preceq_\Psi$ and for conditional belief bases that satisfies (QPCP$_\circ$). Let $\mathcal{R} = \{(B|A)\}$, and let $\Psi^\circ = \Psi \circ \mathcal{R}$. Then for any two worlds ω_1, ω_2 the following conditions are satisfied:*

(**CP1a**) *If $\omega_1, \omega_2 \models AB$ then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi\circ} \omega_2$.*

(**CP1b**) *If $\omega_1, \omega_2 \models A\overline{B}$ then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi\circ} \omega_2$.*

(**CP1c**) *If $\omega_1, \omega_2 \models \overline{A}$ then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi\circ} \omega_2$.*

Proof. If $\omega_1, \omega_2 \models AB$, we have $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2) = \mathbf{a}_1^+$. So, the two sets $\Omega_1 = \{\omega_1\}$ and $\Omega_2 = \{\omega_2\}$ satisfy $\sigma_{\mathcal{R}}(\Omega_1) = \sigma_{\mathcal{R}}(\Omega_2)$, and since both sets contain just one element, $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi\circ} \omega_2$ by (QPCP$_\circ$)(1'), and $\omega_1 \preceq_{\Psi\circ} \omega_2$ implies $\omega_1 \preceq_\Psi \omega_2$ by (QPCP$_\circ$)(2'). This proves (CP1a).

The proofs for (CP1b) and (CP1c) are completely analogous because we have $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ in each case. □

In case of a single proposition $A \equiv (A|\top)$, verification of the conditional coincides with A being true, and falsification means that A is false. These are exactly the first two cases that are addressed by the axioms in (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017) where both propose the same axioms:

Proposition 7 *Let $\circ$ be a change operator for epistemic states Ψ equipped with a total preorder $\preceq_\Psi$ and for single propositions. Then $\circ$ satisfies (CP1a-b) iff it satisfies (DP1-2) resp. (KPP1-2).*

This establishes the first important connection to the works of (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017) and reveals that their first two axioms do not coincide just for technical reasons but that they can be justified by a common change prinicple which is implied by (QPCP$_\circ$).

In order to show that also the other axioms of (Darwiche and Pearl 1997) and (Konieczny and Pino Pérez 2017) are covered by (QPCP$_\circ$), we have to consider the specific cases of iterated revision and contraction which most crucially differ by their success conditions: For revision operators which will be denoted by $*$, beliefs should be adopted, and for contraction operators denoted by $-$, beliefs should be given up. For the case of (sets of) conditional beliefs, they can be adapted in a straightforward way:

(Success*) $\Psi * \mathcal{R} \models \mathcal{R}$, i.e., $\Psi * \mathcal{R} \models (B|A)$ for all $(B|A) \in \mathcal{R}$.

(Success$^-$) $\Psi - \mathcal{R} \not\models \mathcal{R}$, i.e., $\Psi - \mathcal{R} \not\models (B|A)$ for some $(B|A) \in \mathcal{R}$.

We further need to express an idea of minimal change that is most basic to AGM theory and aims at avoiding unnecessary change in case the new beliefs are already present in the prior state:

(Stability*) If $\Psi \models \mathcal{R}$, then $\Psi * \mathcal{R} = \Psi$.

(Stability$^-$) If $\Psi \not\models \mathcal{R}$, i.e., $\Psi \not\models (B|A)$ for some $(B|A) \in \mathcal{R}$, then $\Psi - \mathcal{R} = \Psi$.

(Stability*) has been proposed already in (Kern-Isberner 2001; 2004; Kern-Isberner and Huvermann 2017) for various forms of iterated revision. In general, the stability axioms make clear that the change operators considered here are not to strengthen resp. weaken beliefs. These might be desirable effects of change but indicate change scenarios that are substantially different from those that are considered in this paper. Nevertheless, (QPCP$_\circ$) as a general guidelines for changes may also be relevant for other change scenarios because it is formalized independently of success or stability conditions.

We are now ready to show further important consequences of (QPCP$_\circ$); we denote the respective version of (QPCP$_\circ$) for revision and contraction by (QPCP$_*$) resp. (QPCP$_-$). First, we deal with the case of revision:

Proposition 8 *Let $*$ be an iterated revision operator for epistemic states Ψ and for conditional belief bases that satisfies (QPCP$_*$), (Success*), and (Stability*). Let $\mathcal{R} = \{(B|A)\}$, and let $\Psi^* = \Psi * \mathcal{R}$. Then for any two worlds ω_1, ω_2 the following conditions are satisfied:*

(CP2*) *If $\omega_1 \models AB$ and $\omega_2 \models A\overline{B}$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi*} \omega_2$.*

(CP3*) *If $\omega_1 \models AB$ and $\omega_2 \models A\overline{B}$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi*} \omega_2$.*

Proof. As a general set-up for the proof, let $\omega_1 \models AB$ and $\omega_2 \models A\overline{B}$. Due to (Success*), $\Psi^* \models (B|A)$, and hence $\min_{\Psi*}(AB) \prec_{\Psi*} \min_{\Psi*}(A\overline{B})$. Let $\omega_1' \in \min_{\Psi*}(AB)$ and $\omega_2' \in \min_{\Psi*}(A\overline{B})$, then $\omega_1' \prec_{\Psi*} \omega_2'$. Because of (CP1a-b), the order among models of AB resp. $A\overline{B}$ is maintained under revision, so also $\omega_1' \in \min_\Psi(AB)$ and $\omega_2' \in \min_\Psi(A\overline{B})$. Consider the sets $\Omega_1 = \{\omega_1, \omega_2'\}$ and $\Omega_2 = \{\omega_2, \omega_1'\}$. We have $\omega_1, \omega_1' \models AB$ and $\omega_2, \omega_2' \models A\overline{B}$, so $\sigma_{\mathcal{R}}(\Omega_1) = \mathbf{a}_1^+\mathbf{a}_1^- = \sigma_{\mathcal{R}}(\Omega_2)$.

First, we show (CP2*). Here, we have $\omega_1 \prec_\Psi \omega_2$. According to (QPCP$_*$)(1), $\omega_1 \prec_\Psi \omega_2$ implies (i) $\omega_1 \prec_{\Psi*} \omega_2$, or (ii) $\omega_2' \prec_{\Psi*} \omega_1'$, or (iii) $\omega_1' \prec_\Psi \omega_2'$. In case (i), we are done; case (ii) cannot occur because of $\omega_1' \prec_{\Psi*} \omega_2'$. So we are left with case (iii) where we have $AB \prec_\Psi A\overline{B}$, so $\Psi \models (B|A)$, and by (Stability*), $\Psi^* = \Psi$ which immediately yields $\omega_1 \prec_{\Psi*} \omega_2$.

Regarding (CP3*), let $\omega_1 \preceq_\Psi \omega_2$. By (QPCP$_*$)(1'), we have (i) $\omega_1 \prec_{\Psi*} \omega_2$, or (ii) $\omega_2' \prec_{\Psi*} \omega_1'$, or (iii) $\omega_1 \preceq_{\Psi*} \omega_2$ and $\omega_2' \preceq_{\Psi*} \omega_1'$, or (iv) $\omega_1' \prec_\Psi \omega_2'$. Cases (ii) and (iii) cannot occur due to (Success*). In case (i), we also have $\omega_1 \preceq_{\Psi*} \omega_2$, so we are done. In case (iv), again (Stability*) ensures that $\Psi^* = \Psi$ and hence $\omega_1 \preceq_{\Psi*} \omega_2$. □

A similar result can be proved for iterated contraction:

Proposition 9 *Let $-$ be an iterated contraction operator for epistemic states Ψ and conditional belief bases that satisfies (QPCP$_-$), (Success$^-$), and (Stability$^-$). Let $\mathcal{R} = \{(B|A)\}$, and let $\Psi^- = \Psi - \mathcal{R}$. Then for any two worlds ω_1, ω_2 the following conditions are satisfied:*

(CP2$^-$) *If $\omega_1 \models A\overline{B}$ and $\omega_2 \models AB$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi-} \omega_2$.*

(CP3$^-$) *If $\omega_1 \models A\overline{B}$ and $\omega_2 \models AB$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi-} \omega_2$.*

Proof. The proof of this proposition follows lines very similar to the proof of Proposition 8, with the roles of AB and $A\overline{B}$ interchanged. The general set-up for the proof is as follows: Let $\omega_1 \models A\overline{B}$, $\omega_2 \models AB$. Due to (Success$^-$), $\Psi^- \not\models (B|A)$, and hence $\min_{\Psi-}(A\overline{B}) \preceq_{\Psi-} \min_{\Psi-}(AB)$. Let $\omega_1' \in \min_{\Psi-}(A\overline{B})$ and $\omega_2' \in \min_{\Psi-}(AB)$, then $\omega_1' \preceq_{\Psi-} \omega_2'$. Because of (CP1a-b), the order among models of AB resp. $A\overline{B}$ is maintained under contraction, so also $\omega_1' \in \min_\Psi(A\overline{B})$ and $\omega_2' \in \min_\Psi(AB)$. Consider the sets

$\Omega_1 = \{\omega_1, \omega_2'\}$ and $\Omega_2 = \{\omega_2, \omega_1'\}$. Again, $\sigma_{\mathcal{R}}(\Omega_1) = \mathbf{a}_1^+ \mathbf{a}_1^- = \sigma_{\mathcal{R}}(\Omega_2)$.

Starting with (CP2$^-$), let $\omega_1 \prec_{\Psi} \omega_2$. According to (QPCP$_-$)(1), $\omega_1 \prec_{\Psi} \omega_2$ implies (i) $\omega_1 \prec_{\Psi^-} \omega_2$, or (ii) $\omega_2' \prec_{\Psi^-} \omega_1'$, or (iii) $\omega_1' \prec_{\Psi} \omega_2'$. In case (i), we are done; case (ii) cannot occur because of (Success$^-$). In case (iii), we have $A\overline{B} \prec_{\Psi} AB$, so $\Psi \not\models (B|A)$, and by (Stability$^-$), $\Psi^- = \Psi$ which immediately yields $\omega_1 \prec_{\Psi^-} \omega_2$.

For (CP3$^-$), let $\omega_1 \preceq_{\Psi} \omega_2$. By (QPCP$_-$)(1'), we have (i) $\omega_1 \prec_{\Psi^-} \omega_2$, or (ii) $\omega_2' \prec_{\Psi^-} \omega_1'$, or (iii) $\omega_1 \preceq_{\Psi^-} \omega_2$ and $\omega_2' \preceq_{\Psi^-} \omega_1'$, or (iv) $\omega_1' \prec_{\Psi} \omega_2'$. In cases (i) and (iii), we are done, case (ii) conflicts with (Success$^-$). In case (iv), again (Stability$^-$) ensures that $\Psi^- = \Psi$ and hence $\omega_1 \preceq_{\Psi^-} \omega_2$. $\square$

Now, it can be seen immediately that postulates (DP3-4) are equivalent to (CP2*) and (CP3*) for the case $(A|\top) \equiv A$, as well as that postulates (KPP3-4) are equivalent to (CP2$^-$) and (CP3$^-$) for the same case. Taking also the results from Proposition 7 into account, we obtain one of the main results of this paper:

Theorem 2 *The qualitative principle of conditional preservation (QPCP$_*$) for revision together with (Success*) and (Stability*) imply all postulates (DP1-4) of (Darwiche and Pearl 1997). Moreover, the qualitative principle of conditional preservation (QPCP$_-$) for contraction together with (Success$^-$) and (Stability$^-$) imply all postulates (KPP1-4) of (Konieczny and Pino Pérez 2017).*

So together with the basic axioms of success and stability, the qualitative principle of conditional preservation (QPCP$_\circ$) is able to cover major approaches to iterated revision and contraction previously published, and even is able to provide guidelines for much more general revision problems. It is capable of axiomatizing the idea of conditional preservation much more deeply than (Darwiche and Pearl 1997) by making use of conditional structures which can truely handle (sets of) conditionals.

(QPCP$_\circ$) has been shown to have a good theoretical impact by bringing together major previous approaches to iterated belief change, and opening the way to conditional belief change even for qualitative epistemic states. For practical problems and illustrations, however, the approach of c-change operators as characterized in Theorem 1 is usually much more intuitive and easier to understand. Numerous examples of c-revisions can be found in (Kern-Isberner 2001; 2004; Kern-Isberner and Huvermann 2017) and related works. Therefore, we illustrate c-contractions and the idea of conditional preservation in the following example.

Example 3 *Consider the conditional belief base $\mathcal{R} = \{(f|b), (b|p)\}$ containing the conditionals $(f|b)$ "birds (usually) fly" and $(b|p)$ "penguins are birds". Table 1 shows an OCF κ that accepts $\mathcal{R}$.*

As one might have expected, $\kappa \models (f|p)$ holds, i.e., penguins are believed to be flying objects, but κ is indifferent with respect to penguins and birds since $\kappa \not\models p, \overline{p}, b, \overline{b}$. We want to forget that penguins can fly which can be done via a contraction $\kappa - (f|p) = \kappa^-$. Making use of $(PCP_\circ^{ocf})$ resp. the schema given by Theorem 1 with appropriate parame-

ω	$\kappa(\omega)$	ω	$\kappa(\omega)$
$p\,b\,f$	0	$\overline{p}\,b\,f$	0
$p\,b\,\overline{f}$	1	$\overline{p}\,b\,\overline{f}$	1
$p\,\overline{b}\,f$	1	$\overline{p}\,\overline{b}\,f$	0
$p\,\overline{b}\,\overline{f}$	1	$\overline{p}\,\overline{b}\,\overline{f}$	0

Table 1: An OCF model of $\mathcal{R}$ in Example 3

ω	$\kappa(\omega)$	$\kappa^-(\omega)$	$\kappa^{-,-}$
$p\,b\,f$	0	1	0
$\overline{p}\,b\,f$	0	0	0
$p\,b\,\overline{f}$	1	1	0
$\overline{p}\,b\,\overline{f}$	1	1	1
$p\,\overline{b}\,f$	1	2	1
$\overline{p}\,\overline{b}\,f$	0	0	0
$p\,\overline{b}\,\overline{f}$	1	1	0
$\overline{p}\,\overline{b}\,\overline{f}f$	0	0	0

Table 2: OCFs κ, κ^- and $\kappa^{-,-}$ for Example 3.

ters, we obtain

$$\kappa - (f|p)(\omega) = \kappa(\omega) + \begin{cases} 1 \text{ if } \omega \models pf \\ 0 \text{ if } \omega \models p\overline{f} \\ 0 \text{ if } \omega \models \overline{p} \end{cases}$$

as a suitable c-contraction. Here, we find the new conditional belief $\kappa^- \models (\overline{f}|p\overline{b})$ which was not present in κ ($\kappa \not\models (\dot{f}|p\overline{b})$). Still κ is indifferent to birds ($\kappa^- \not\models \dot{b}$), but now $\kappa^- \models \overline{p}$ – penguins are believed to be exceptional in κ^- which makes perfect sense.

However, assume that now we want to forget that penguins are exceptional:

$$\kappa^{-,-} = \kappa^- - \overline{p}.$$

The c-contraction $\kappa^{-,-}$ can do the job:

$$\kappa^{-,-}(\omega) = \kappa^-(\omega) + \begin{cases} 0 \ \ \text{if } \omega \models \overline{p} \\ -1 \text{ if } \omega \models p \end{cases}$$

As expected, we find $\kappa^{-,-} \not\models \dot{p}, \dot{b}$, as in κ; however, also $(f|b), (b|p)$ have been forgotten – $\kappa^{-,-} \not\models (\dot{f}|b), (\dot{b}|p)$. Of course, we have contracted beliefs which have been induced by our first belief base $\mathcal{R}$, raising doubts about the conditionals in $\mathcal{R}$ in this way, which is reflected by $\kappa^{-,-}$. But still we expect to find some traces of the conditionals in $\mathcal{R}$ under the paradigm of conditional preservation. And indeed, as can easily be seen from Table 2, $\kappa^{-,-}$ still remembers the original conditionals, but in a refined form:

$$\kappa^{-,-} \models (f|b\overline{p}) \text{ and } \kappa^{-,-} \models (b|pf)$$

This example illustrates how interactions among conditional beliefs in the prior and the posterior state and between those states are taken into account and dealt with in a careful way by the principle of conditional preservation.

6 Conclusion and Future Work

In this paper, we presented a purely qualitative version of the powerful principle of conditional preservation for iterated belief change that has brought forth solutions to many important problems in belief change theory (Darwiche and Pearl 1997; Konieczny and Pino Pérez 2017; Jin and Thielscher 2007; Delgrande and Jin 2012). Indeed, we showed that this principle is able to provide the missing link connecting major works in iterated belief change theory. Remarkably, although making use only of comparisons between worlds, the principle may help solving advanced belief change problems such as multiple and conditional belief change, and combinations thereof.

For future work, we will apply this novel qualitative principle to more problems in iterated belief change that have been solved successfully by its semi-quantitative archetype for OCFs (Kern-Isberner 2004; Kern-Isberner et al. 2017), in particular, we will focus on independence properties in multiple belief change (Jin and Thielscher 2007; Delgrande and Jin 2012; Kern-Isberner and Huvermann 2017).

Of course, having a syntactic counterpart to $(QPCP_\circ)$ and a representation theorem would be desirable. However, conditional interactions can be extremely complicated on the syntactic level, so a representation theorem will be very hard to set up, and then might only provide few insights and little benefit, in particular, when compared to the clear representation Theorem 1. It might be more useful to derive further postulates for iterated (conditional) belief change in ways that are similar to the proofs of Propositions 8 and 9.

References

Alchourrón, C.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50(2):510–530.

Chopra, S.; Ghose, A.; Meyer, T.; and Wong, K.-S. 2008. Iterated belief change and the recovery axiom. *Journal of Philosophical Logic* 37:501–520.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial Intelligence* 89:1–29.

DeFinetti, B. 1974. *Theory of Probability*, volume 1,2. New York: John Wiley and Sons.

Delgrande, J. P., and Jin, Y. 2012. Parallel belief revision: Revising by sets of formulas. *Artificial Intelligence* 176(1):2223–2245.

Eichhorn, C.; Kern-Isberner, G.; and Ragni, M. 2018. Rational inference patterns based on conditional logic. In McIlraith, S., and Weinberger, K., eds., *Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence, (AAAI-18)*. AAAI Press.

Fine, B., and Rosenberger, G. 1999. *Algebraic Generalizations of Discrete Groups*. New York, Basel: Dekker.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171(1):1–18.

Kern-Isberner, G., and Huvermann, D. 2017. What kind of independence do we need for multiple iterated belief change? *J. Applied Logic* 22:91–119.

Kern-Isberner, G., and Krümpelmann, P. 2011. A constructive approach to independent and evidence retaining belief revision by general information sets. In Walsh, T., ed., *Proceedings 22nd International Joint Conference on Artificial Intelligence, IJCAI'2011*, 937–942. Menlo Park, CA: AAAI Press.

Kern-Isberner, G.; Bock, T.; Sauerwald, K.; and Beierle, C. 2017. Iterated contraction of propositions and conditionals under the principle of conditional preservation. In Benzmüller, C.; Lisetti, C.; and Theobald, M., eds., *Proceedings 3rd Global Conference on Artificial Intelligence, GCAI 2017*, volume 50, 78–92. EPiC Series in Computing.

Kern-Isberner, G. 1998. Characterizing the principle of minimum cross-entropy within a conditional-logical framework. *Artificial Intelligence* 98:169–208.

Kern-Isberner, G. 1999. Postulates for conditional belief revision. In *Proceedings Sixteenth International Joint Conference on Artificial Intelligence, IJCAI-99*, 186–191. Morgan Kaufmann.

Kern-Isberner, G. 2001. *Conditionals in nonmonotonic reasoning and belief revision*. Springer, Lecture Notes in Artificial Intelligence LNAI 2087.

Kern-Isberner, G. 2004. A thorough axiomatization of a principle of conditional preservation in belief revision. *Annals of Mathematics and Artificial Intelligence* 40(1-2):127–164.

Konieczny, S., and Pino Pérez, R. 2017. On iterated contraction: syntactic characterization, representation theorem and limitations of the levi identity. In Moral, S.; Pivert, O.; Sánchez, D.; and Marín, N., eds., *Proceedings of the 11th International Conference of Scalable Uncertainty Management, SUM 2017*, volume 10564 of *Lecture Notes in Computer Science*, 348–362. Springer.

Ramachandran, R.; Nayak, A.; and Orgun, M. 2012. Three approaches to iterated belief contraction. *Journal of Philosophical Logic* 41:115–142.

Spohn, W. 1988. Ordinal conditional functions: a dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics, II*. Kluwer Academic Publishers. 105–134.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford University Press.