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Default Reasoning via Topology and Mathematical Analysis: A Preliminary Report

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Abstract

A default consequence relation $\alpha \mid\!\sim \beta$ (if α, then *normally* β) can be naturally interpreted via a '*most*' generalized quantifier: $\alpha \mid\!\sim \beta$ is valid iff in 'most' α-worlds, β is also true. We define various semantic incarnations of this principle which attempt to make the set of $(\alpha \wedge \beta)$-worlds 'large' and the set of $(\alpha \wedge \neg\beta)$-worlds 'small'. The straightforward implementation of this idea on finite sets is via '*clear majority*'. We proceed to examine different '*majority*' interpretations of normality which are defined upon notions of classical mathematics which formalize aspects of '*size*'. We define default consequence using the notion of *asymptotic density* from analytic number theory. Asymptotic density provides a way to measure the size of integer sequences in a way much more fine-grained and accurate than set cardinality. Further on, in a topological setting, we identify '*large*' sets with *dense sets* and '*negligibly small*' sets with *nowhere dense* sets. Finally, we define default consequence via the concept of *measure*, classically developed in mathematical analysis for capturing '*size*' through a generalization of the notions of *length*, *area* and *volume*. The logics defined via *asymptotic density* and *measure* are weaker than the KLM system $\mathbf{P}$, the so-called '*conservative core*' of nonmonotonic reasoning, and they resemble to probabilistic consequence. Topology goes a longer way towards system $\mathbf{P}$ but it misses *Cautious Monotony* ($\mathbf{CM}$) and $\mathbf{AND}$. Our results show that a 'size'-oriented interpretation of default reasoning is context-sensitive and in 'most' cases it departs from the preferential approach.

1 Introduction

The study of nonmonotonic consequence relations was initiated with Gabbay's pioneer work (Gabbay 1985) and reached a certain level of maturity and sophistication with the landmark work(s) of Kraus, Lehmann and Magidor (KLM) (Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992). After a decade of investigations in nonmonotonic reasoning (NMR) which had resulted in interesting logical frameworks (*Default Logic*, *Autoepistemic Logic*, *Circumscription*, etc.), but not in a unifying theory to serve as a global benchmark, the classification of nonmonotonic consequence relations achieved in the KLM framework was an important milestone.

The KLM framework focuses on the study of the fundamental properties of the consequence relation $\alpha \mid\!\sim \beta$, which is read as: 'if α is true, then *normally* (*by default*) β should also be true'. This object of the metalanguage ($\mid\!\sim$) is also called a *default conditional* and it is not unusual to encounter other readings of it, including 'if α then *typically* β' or 'α *generally* (*usually*) implies β'. The KLM study of abstract properties of consequence relations resulted in a hierarchy of systems with increasing inferential strength: the *Cumulative Logics* (system $\mathbf{C}$), the *Loop-Cumulative Logics* (system $\mathbf{CL}$), the *Preferential Logics* (system $\mathbf{P}$) and the *Rational Logics* (system $\mathbf{R}$). The KLM framework became the 'industry standard' in the study of metatheoretic principles of NMR and an important tool in the study of nonmonotonic inference (Makinson 2005). There exists a remarkable consensus on the set of inferential principles which '*should*' be available in any system of commonsense reasoning; this is the import of the fact that diverse conditional approaches - besides KLM, based upon different intuitions, essentially resulted to the same set of inferential principles: this includes approaches based on *probability theory* (Pearl 1988, ϵ-entailment), (Adams 1975, p-entailment), *possibilistic logic* (Benferhat, Dubois, and Prade 1997; Dubois and Prade 1991) and *preferential reasoning* (Geffner and Pearl 1992)[1]. Thus, it has been argued that the corresponding set of inferential principles should be considered as the '*conservative core*' common to all default inference systems: this denotes any **supraclassical**, **nonmonotonic** system, satisfying $\mathbf{REF}$, $\mathbf{LLE}$, $\mathbf{RW}$, $\mathbf{CUT}$, $\mathbf{CM}$, $\mathbf{AND}$, $\mathbf{OR}$, that is (a logic matching) the KLM system $\mathbf{P}$. Despite its widespread acceptance however, this perspective is not problem-free: it is well-known that $\mathbf{AND}$ is invalidated in *supraclassical probabilistic consequence* (Makinson 2005, p. 128) while other probabilistic approaches also result in weaker logics. For a recent discussion on the appropriateness of $\mathbf{P}$ as the the 'conservative core' of NMR see (Delgrande and Renne 2016).

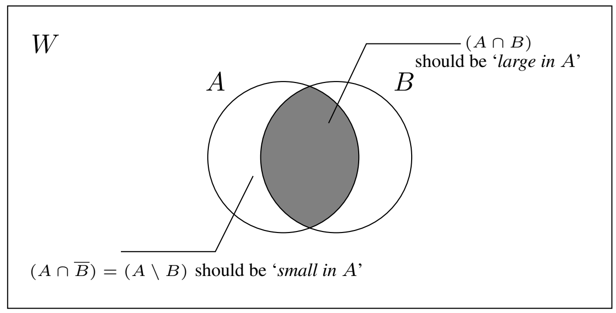
[1]The reader is also referred to the (Eiter and Lukasiewicz 2000) paper which is one of the authors' favourites. It discusses the complexity of default reasoning from conditional knowledge bases but in doing so, it provides a concise and readable account of different approaches to the semantics of default conditionals.

The KLM investigations were mainly syntactic but a semantic counterpart, inspired from Shoham's preferential semantics (Shoham 1987), almost readily emerged: the KLM models are possible-worlds structures endowed with a preference relation between worlds. Interestingly enough, the study of the abstract properties of nonmonotonic consequence was quickly connected to the investigations on *Conditional Logics* (Kern-Isberner 2001; Delgrande 1987) and it was revealed that the KLM logics correspond to the 'flat' fragment of well-known conditional logics. Aspects of the KLM framework have been successfully injected in areas of logic-based Knowledge Representation (Britz and Varzinczak 2016; 2017; Beierle and Kern-Isberner 2012).

Still, after thirty five years of NMR, the interpretation of a '*default*', that is, a statement of the form 'if α, then normally β' is far from being clear. For a comprehensive analysis, we refer the reader to J. Delgrande's paper in the **NonMon@30 workshop**, which celebrated thirty years from the 'birth' of the field, typically set in 1980, the year of the seminal publications which initiated the study of nonmonotonic logics (Delgrande 2012, '*What's in a Default?*'). Several 'readings' of a default are possible, including a well-known '*preferential*' one and an '*epistemic*' one. What is of our concern in this paper is a **'majority' interpretation of defaults**: interpreting $\alpha \mid\!\sim \beta$ as 'if α is true, then in '*most*' cases β is true' - '*if it is a rainy day, then in most cases you will find a traffic jam*'. So, the question really transforms into how to define correctly a '*most*' generalized quantifier which will properly interpret default statements.

The study of *generalized quantifiers* is an old art; it has been an active field of research inside the model theory of mathematical logic (Bell and Slomson 1969, Ch. 13) where set-theoretic variants of a 'most' quantifier have been investigated. In KR, 'size'-oriented approaches to default reasoning have appeared in (Schlechta 1995, '*Defaults as Generalized Quantifiers*') and a recent account can be found in (Gabbay and Schlechta 2011). A study of *majority default conditional logic* can be also found in (Jauregui 2008). These approaches identify '*large subsets*' of a set (corresponding to a '*most*' quantifier) with the elements of a so-called *weak filter*. Assuming a universe W, a **weak filter** is a non-empty collection of subsets of W which, (i) is upwards closed (any superset of a 'large' set should be large) and (ii) contains only pairwise non-disjoint sets (two 'large' sets should overlap). This is a 'relaxation' of classical filters but still suffers from the defects of the classical setting. The class of weak filters properly contains the (classical) filters, and there exist filters (the *principal* ones) which may contain finite sets, even singletons (if they are maximal), which we would be reluctant to call '*large subsets*'. In fairness to the weak-filter approach to 'majority' we should mention that the two properties of 'weak filters' are just the 'minimum' requirements one should expect from 'large' sets and do not claim to pin down what a '*collection of large subsets*' is.

In this paper we reconsider the interpretation of default consequence via a '*most*' generalized quantifier, employing different *analytic* or *topological* incarnations of '*largeness*'. The starting point is that the $(\alpha \wedge \beta)$-worlds should correspond to a '*large subset*' of the α-worlds and the $(\alpha \wedge \neg\beta)$-worlds to a '*small*' one: "*a natural interpretation of the nonmonotonic rule $\phi \mid\!\sim \psi$ is that the set of exceptional cases, i.e. those where ϕ holds but not ψ, is a small subset of all the cases where ϕ holds, and the complement, i.e. the set of cases where ϕ and ψ hold, is a big subset of all ϕ-cases*" (Gabbay and Schlechta 2011, Ch. 5.1.2, p. 153). Let us grasp the opportunity to emphasize that the one component of this 'requirement' does not necessarily imply the other. Within model theory, a collection of 'big subsets' is almost a synonym for an *ultrafilter*. Consider an ultrafilter over $\mathbb{N}$: it will contain either the set of even natural numbers or the set of odd natural numbers, but not both. It seems a bit 'paradoxical' to accept one of them as 'large' and the other as 'small'.



The approach we follow is actually a 'size'-oriented adaptation of the dominant view on conditional semantics (Bochman 2001). To proceed with the 'majority' interpretation of defaults, the most straightforward interpretation of 'largeness' is via set cardinality and it seems to work smoothly if the set of situations is finite. A '*clear*' majority is the obvious answer and this is investigated in Section 3. When jumping to the infinite, the situation becomes complicated and cardinality is provably insufficient. Thus, we proceed to use *analytic tools* for defining '*large*' and '*small*' sets. In Section 4 we work with the set $\mathbb{N}$ of natural numbers and we identify '*size*' of sets via the notion of *asymptotic density* of integer sequences; intuitively, this method represents a more precise way to capture the size of sets of natural numbers. Moving to sets of arbitrary cardinality, we work in Section 5 with topological notions. *Topology* is a very flexible tool and there exists a natural topological encoding of notions like '*closeness*', '*smallness*', '*largeness*' and '*thickness*'. We identify 'large sets' with the (everywhere) '*dense sets*' and the 'small sets' with the '*nowhere dense sets*'; the emerging models are somewhat complicated and the logics approach the '*conservative core*' but fail to satisfy Cautious Monotony ($\mathbf{CM}$) and $\mathbf{AND}$. Finally, in Section 6 we employ the concept of *measure* which captures 'size' (or 'mass') in a different way, closely related to probability.

We discuss the results and possible future work, inside each section. We conclude in Section 7 with a brief discussion on similar results. We should defend our excursion to *Topology* and *Mathematical Analysis*: it is true that a rational agent's universe is finite but the fundamental studies in logic have always benefited from excursions to the infinite and it is amazing that parts of the 'highest infinite' have concrete effects at the level of the natural numbers.

2 Notation and Terminology

Going through the results of this paper requires material usually falling under the heading(s) of **Analytic Number Theory** (*asymptotic density* - Section 4), **Real Analysis** (standard *topology of* $\mathbb{R}$ - Section 5, *limits* of sequences ($\lim$, $\liminf$, $\limsup$) - Section 4), **Mathematical Analysis** (*measure theory* - Section 6) and **Point-Set Topology** (**General Topology**, Section 5). Instead of providing a 'Background' section, we have opted for a review of the necessary notions and facts inside each section as needed; references are also provided therein. Regarding the theory of *default consequence*, we assume that the reader is acquainted with the study of nonmonotonic consequence relations and, in particular, the basic facts about KLM logics. The inferential principles we have investigated can be found in Table 1 which succintly summarizes the profile of the logics we obtained, against the KLM logics, but not limited to it. We have followed the style and terminology of the original KLM papers; the reader is also referred to the references provided in the introductory Section 1. In Table 1: a ✓denotes that the corresponding inferential principle is validated in the models of the corresponding logic, a ✗ denotes that it is invalidated, and a – – that its status is unknown for the time being. Due to space limitations, we provide fragments of the proofs and we leave the rest of the arguments for the full report.

We assume a *propositional language* with a countably infinite set of propositional variables. The basis for the **models** $\mathcal{M} = \langle W, V \rangle$ of our majority-default logics is simple: a set W of *possible worlds* (*states*, *points*) and a valuation V which assigns a set of points to each propositional variable. In a standard fashion, V extends to a valuation $\overline{V}$ over the whole set of formulas of the language. $\overline{V}(\beta)$ is also called the truth set $\|\beta\|_{\mathcal{M}}$ of β in $\mathcal{M}$. To simplify the notation we follow the convention of denoting formulas with lowercase Greek letters $\alpha, \beta, \gamma, \dots$ and identifying their truth sets with '*corresponding*' uppercase Latin letters: $\|\alpha\|_{\mathcal{M}} = A$, $\|\beta\|_{\mathcal{M}} = B$, $\|\gamma\|_{\mathcal{M}} = C$ and so on. We will use fairly standard notation for the sets $\mathbb{R}$ (reals), $\mathbb{Q}$ (rationals), $\mathbb{Z}$ (integers) and $\mathbb{Z}^+$, $\mathbb{N}$ (the set of natural numbers: $\{0, 1, 2, 3, \dots\}$) and $\mathbb{N}^*$ for positive integers ($\mathbb{N} \setminus \{0\}$).

Evaluation of a default conditional is **global** and not local: we examine the validity of $\alpha \mid\!\sim \beta$ in $\mathcal{M}$ and not its truth at a point w inside $\mathcal{M}$; this is an essential part of our 'majority' approach. One can, in principle, define local majority evaluation, cf. the (weak) filter-based neighborhood semantics in (Jauregui 2008), but this is not the road taken in this paper. Note also that default consequence $\alpha \mid\!\sim \beta$ is treated as an element of the metalanguage, in the standard tradition of nonmonotonic consequence relations: nesting is not allowed for default conditionals. Finally, our models will take the form $\mathcal{M} = \langle \mathbb{N}^*, V \rangle$ in Section 4, the form $\mathcal{M} = \langle \langle W, \tau \rangle, V \rangle$, where τ is a *topology* on W in Section 5 and $\mathcal{M} = \langle \langle W, \mu \rangle, V \rangle$, where μ is a *measure* on the powerset algebra of W in Section 6.

3 Clear Majority on Finite Sets of Points

The most straightforward and familiar approach to a '*most*' quantifier is via '*clear majority*' on a finite set of states. Assume a finite universe W and interpret $\alpha \mid\!\sim \beta$ as a denotation of the fact that the number of $(\alpha \wedge \beta)$-states clearly exceeds the number of the $(\alpha \wedge \neg\beta)$-ones. To simplify the overloaded notation, we will 'locally' denote by $\#\|\alpha\|_{\mathcal{M}}$ the number of α-worlds in $\mathcal{M}$.

Definition 3.1 Assume a model $\mathcal{M} = \langle W, V \rangle$, W finite. Then, $\alpha \mid\!\sim \beta$ is valid in $\mathcal{M}$ iff either $\|\alpha\|_{\mathcal{M}} = \emptyset$ or

$$\#\|(\alpha \wedge \neg\beta)\|_{\mathcal{M}} \; < \; \#\|(\alpha \wedge \beta)\|_{\mathcal{M}}$$

Assuming the convention we introduced in Section 2, the condition is simply written as: $|A \setminus B| \; < \; |A \cap B|$, or equivalently, $|A \cap \overline{B}| \; < \; |A \cap B|$ ($\mathcal{M}$ is implicit from context). Following a standard tradition in logical investigations on generalized quantifiers, we have interpreted '*clear majority*' as '*simple majority*'. Actually, as we show in the full report, this choice is inconsequential. The following theorem gathers the properties of the default conditional erected over Def. 3.1.

Theorem 3.2 The logic of Definition 3.1 satisfies the axioms and rules denoted in the appropriate column of Table 1.

4 Asymptotic Density of Integer Sequences

Moving to infinite sets of possible worlds, clearly calls for a careful reconsideration of the notion of 'majority'. Assume the set $\mathbb{N}$ of the natural numbers, the *first infinite ordinal* naturally constructed in *Set Theory* with the *Axiom of Infinity*. From the cardinality viewpoint, the set of *even natural numbers*, the set of *perfect squares*, the set of multiples of 10^{50}, they are all equipollent, sharing the cardinality $\aleph_0$ of $\mathbb{N}$. However, we would be probably willing to accept the intuitive validity of informal statements like '*half of the positive integers are even*' or '*there exist more even positive integers than perfect squares*' and '*perfect squares become increasingly 'scarce' as we proceed through the positive integers*'. These statements are meaningless in the set-theoretic context but can be made a bit more precise when cast in the form of a statement like '*if a natural number is randomly chosen, it will be even with 'probability'* $1/2$' or '*almost no positive integer can be written as the sum of two squares*'. The natural candidate for a formalization of these intuitive statements is *probability theory* but it clearly fails: it can be shown that any probability measure on $\mathbb{N}^*$ violates two natural properties we expect from such a formalization: (i) the probability of the set of multiples of k ($k \in \mathbb{N}^*$) should be $1/k$, and (ii) any two sets of positive integers, with finite symmetric difference (which means that they are 'almost' identical) should be assigned the same probability (Tenenbaum 2015, Chapter III.1, Th. 1.1, p. 413).

At this point, techniques from *Mathematical Analysis* come into play. The **asymptotic density** of integer sequences is a notion developed in *analytic number theory* (Tenenbaum 2015; de Koninck and Luca 2012) with the aim of providing a precise formalization of the intuitive statements mentioned previously. Given any set $X \subseteq \mathbb{N}^*$, it is clear that the probability that a randomly chosen number from the interval $[1, n]$ belongs to X is $\frac{|X \cap [1, n]|}{n}$. The limit of this quantity as n tends to infinity, assuming it exists, is called the **asymptotic** (or **natural**) **density** of A:

$$\mathsf{d}(A) = \lim_{n\to\infty} \frac{|A \cap [1,n]|}{n}$$

This limit does not always exist and thus a drawback of asymptotic density is that it is not defined for all $X \subseteq \mathbb{N}^*$: see (de Koninck and Luca 2012, Ch. 1.8, p. 11) for an example. On the other hand, the density is undefined only for some 'pathological' sets; for many useful integer sequences, the asymptotic density exists[2]. The latter is definitely a bit of good news for the number theorists but not a relief for us, since we need a notion which is 'everywhere' defined. Thus, we resort to the 'careful' definition of density, which starts with the **lower** and the **upper density** of X respectively defined as[3]:

$$\mathsf{d}_*(X) = \liminf_{n\to\infty} \frac{|X \cap [1,n]|}{n} \qquad \mathsf{d}^*(X) = \limsup_{n\to\infty} \frac{|X \cap [1,n]|}{n}$$

The upper and the lower density of a set X always exist and the asymptotic density $\mathsf{d}(X)$ exists iff they are equal, in which case $\mathsf{d}_*(X) = \mathsf{d}^*(X) = \mathsf{d}(X)$. The following **properties of asymptotic density** support its role as a 'measure' of the 'size' of integer sequences:

- For any $X \subseteq \mathbb{N}^*$: $\mathsf{d}(X) = 1 - \mathsf{d}(\mathbb{N}^* \setminus X)$, assuming $\mathsf{d}(X)$ exists. Moreover, $\mathsf{d}(\mathbb{N}^*) = 1$ and for finite $X \subseteq \mathbb{N}^*$: $\mathsf{d}(X) = 0$.
- '*Sparse*' sets like the set of *perfect squares* have zero density. The set of *prime numbers* is also of zero density (a direct consequence of the *Prime Number Theorem*).
- The asymptotic density of the set of *odd* (and the set of *even*) positive integers is 1/2.
- For any *arithmetic progression* $A = \{an + b \mid n \in \mathbb{N}^*\} = \{m \mid m \equiv b\,(mod\,a)\}$, it holds that $\mathsf{d}(A) = 1/a$.
- *Monotonicity of density*: assuming existence of the densities, $A \subseteq B \subseteq \mathbb{N}^*$ implies $\mathsf{d}(A) \leq \mathsf{d}(B)$.

The reader is referred to (de Koninck and Luca 2012, Ch. 7.5) for examples of asymptotic densities and to (Tenenbaum 2015, Ch. III.1) for a detailed exposition and definition of other densities which play a fundamental role in the development of *analytic number theory* and *ergodic theory*. Having described a measure of size in (subsets of) $\mathbb{N}^*$, we are ready to study 'majority'-default consequence over $\mathbb{N}^*$. We recall that our models, inside this section, are of the form $\mathcal{M} = \langle \mathbb{N}^*, V \rangle$. Having in mind the intuition described in Section 1 (check also the Figure in Section 1), one might be tempted to define $(\alpha \mathrel{|\!\sim} \beta)$ by making $(A \cap B)$ 'large' (density one) or making the set $(A \cap \overline{B})$ 'small' (zero density). Both approaches fail from the outset: the former invalidates **REF** and the latter validates monotonicity.

Fact 4.1 Set $(\alpha \mathrel{|\!\sim} \beta)$ iff $\mathsf{d}(A \cap \overline{B}) = 0$. The emerging consequence relation is monotonic.

PROOF. Assume that the densities exist. By the antecedent of the rule, $\mathsf{d}(A \cap \overline{B}) = 0$. $(A \cap C \cap \overline{B}) \subseteq (A \cap \overline{B})$ which implies $\mathsf{d}(A \cap C \cap \overline{B}) \leq \mathsf{d}(A \cap \overline{B}) = 0$ (by monotonicity of asymptotic density). ∎

Instead of trying to make $(A \cap B)$ (globally) '*large*' and/or $(A \cap \overline{B})$ (globally) 'small', we will use the machinery of asymptotic density to make $(A \cap B)$ 'large in A', following the intuition described in (Gabbay and Schlechta 2011). We avoid issues on the existence of limits and walk on the (mathematically) safe side by defining the 'largeness' condition using $\liminf$. The following definition is an 'infinitary' version of '*clear majority*' in the asymptotic density setting. We recall that $\|\alpha\|_{\mathcal{M}} = A$, $\|\beta\|_{\mathcal{M}} = B$ and for $X \subseteq \mathbb{N}^*$, set $X_n = (X \cap [1,n])$.

Definition 4.2 Assume a model $\mathcal{M} = \langle \mathbb{N}^*, V \rangle$: $(\alpha \mathrel{|\!\sim} \beta)$ is valid in $\mathcal{M}$ iff

$$\liminf_{n\to\infty} f(A,B,n) > \frac{1}{2}$$

where $f(A,B,n) = \begin{cases} \dfrac{|B_n \cap A_n|}{|A_n|} & \text{if } |A_n| \neq 0 \\ 1 & \text{if } |A_n| = 0 \end{cases}$

Theorem 4.3 The logic of Definition 4.2 is a *supraclassical*, *nonmonotonic* logic which validates **REF**, **LLE**, and **RW** and invalidates the principles denoted in the appropriate column of Table 1.

PROOF. We provide the proof for **RW**. To facilitate exposition, let $x_n = \dfrac{|A_n \cap C_n|}{|C_n|}$ and $y_n = \dfrac{|B_n \cap C_n|}{|C_n|}$. It is easy to verify that both sequences (x_n) and (y_n) are bounded above by 1. By the premises of the rule: $\models (\alpha \to \beta) \implies (A \subseteq B) \implies (A_n \subseteq B_n) \implies (A_n \cap C_n) \subseteq (B_n \cap C_n) \implies |(A_n \cap C_n)| \leq |(B_n \cap C_n)| \implies x_n \leq y_n, \forall n \in \mathbb{N}^*$. It follows that we can write $y_n = x_n + z_n$, for $z_n \geq 0$. Again, the premises of the rule certify that $\liminf_{n\to\infty} x_n > \frac{1}{2}$ and by the superadditivity of limit inferior (and given that for all the sequences involved the $\liminf$ is a real number):

$$\liminf_{n\to\infty} y_n = \liminf_{n\to\infty}(x_n + z_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} z_n > \frac{1}{2}$$

Hence, by Def. 4.2, $\gamma \mathrel{|\!\sim} \beta$ is valid in $\mathcal{M}$. ∎

5 Default Consequence via Topology

The use of *asymptotic density* from the previous section is certainly an advance compared to the '*clear majority*' or coarse methods like the cofinite/finite approach to the '*large*

[2] It can be proved that for every infinite set $A \subseteq \mathbb{N}^*$ such that $\sum_{a\in A} \frac{1}{a} < \infty$, the set of multiples of A has an asymptotic density (Nathanson 1999, Th. 7.14, p. 257).

[3] The definitions use the notions of $\liminf$ and $\limsup$ from *Real Analysis*. These subtle concepts allow us to reduce convergence and limits of real sequences to the same questions about monotone sequences, and - unlike $\lim$ - they exist for every bounded sequence of real numbers. A sequence converges if and only if its $\liminf$ and $\limsup$ coincide, in which case its limit is their common value. The reader is referred to (Hunter 2014, Ch. 3.6) which contains a detailed exposition and several examples.

	KLM			Majority Default Consequence			
Rules and Axioms	**C**	**P**	**R**	**Simple Majority** [Def. 3.1]	**Asymptotic Density** [Def. 4.2]	**Topological Density** [Def. 5.2]	**Measure Theory** [Def. 6.3]
Supraclassicality $\frac{\alpha\models\beta}{\alpha\mid\!\sim\beta}$	✓	✓	✓	✓	✓	✓	✓
REF $\alpha\mid\!\sim\alpha$	✓	✓	✓	✓	✓	✓	✓
LLE $\frac{\models\alpha\equiv\beta \quad \alpha\mid\!\sim\gamma}{\beta\mid\!\sim\gamma}$	✓	✓	✓	✓	✓	✓	✓
RW $\frac{\models\alpha\rightarrow\beta \quad \gamma\mid\!\sim\alpha}{\gamma\mid\!\sim\beta}$	✓	✓	✓	✓	✓	✓	✓
CUT $\frac{\alpha\mid\!\sim\beta \quad (\alpha\wedge\beta)\mid\!\sim\gamma}{\alpha\mid\!\sim\gamma}$	✓	✓	✓	✗	✗	✓	✗
CM $\frac{\alpha\mid\!\sim\beta \quad \alpha\mid\!\sim\gamma}{(\alpha\wedge\beta)\mid\!\sim\gamma}$	✓	✓	✓	✗	✗	✗	✗
AND $\frac{\alpha\mid\!\sim\beta \quad \alpha\mid\!\sim\gamma}{\alpha\mid\!\sim(\beta\wedge\gamma)}$	✓	✓	✓	✗	✗	✗	✗
OR $\frac{\alpha\mid\!\sim\gamma \quad \beta\mid\!\sim\gamma}{(\alpha\vee\beta)\mid\!\sim\gamma}$	✗	✓	✓	✗	✗	✓	✗
RM $\frac{\alpha\mid\!\sim\gamma \quad \alpha\not\mid\!\sim\neg\beta}{(\alpha\wedge\beta)\mid\!\sim\gamma}$	✗	✗	✓	✗	✗	– –	✗
Monotonicity $\frac{\alpha\mid\!\sim\beta}{(\alpha\wedge\gamma)\mid\!\sim\beta}$	✗	✗	✗	✗	✗	✗	✗
Left Strengthening $\frac{\models\alpha\rightarrow\beta \quad \beta\mid\!\sim\gamma}{\alpha\mid\!\sim\gamma}$	✗	✗	✗	✗	✗	✗	✗
Contraposition $\frac{\alpha\mid\!\sim\beta}{\neg\beta\mid\!\sim\neg\alpha}$	✗	✗	✗	✗	✗	✗	✗
Transitivity $\frac{\alpha\mid\!\sim\beta \quad \beta\mid\!\sim\gamma}{\alpha\mid\!\sim\gamma}$	✗	✗	✗	✗	✗	✗	✗
EHD $\frac{\alpha\mid\!\sim(\neg\beta\vee\gamma)}{(\alpha\wedge\beta)\mid\!\sim\gamma}$	✗	✗	✗	✗	✗	✗	✗
Negation Rationality $\frac{(\alpha\wedge\gamma)\not\mid\!\sim\beta \quad (\alpha\wedge\neg\gamma)\not\mid\!\sim\beta}{\alpha\not\mid\!\sim\beta}$	✗	✗	✗	✓	– –	– –	✓
Disjunctive Rationality $\frac{\alpha\not\mid\!\sim\gamma \quad \beta\not\mid\!\sim\gamma}{(\alpha\vee\beta)\not\mid\!\sim\gamma}$	✗	✗	✓	✗	– –	– –	✗
HHD $\frac{(\alpha\wedge\beta)\mid\!\sim\gamma}{\alpha\mid\!\sim(\beta\rightarrow\gamma)}$	✓	✓	✓	✗	– –	– –	✓
D $\frac{(\alpha\wedge\neg\beta)\mid\!\sim\gamma \quad (\alpha\wedge\beta)\mid\!\sim\gamma}{\alpha\mid\!\sim\gamma}$	✗	✓	✓	✓	– –	– –	✓
Equivalence $\frac{\alpha\mid\!\sim\beta \quad \beta\mid\!\sim\alpha \quad \alpha\mid\!\sim\gamma}{\beta\mid\!\sim\gamma}$	✓	✓	✓	✗	✗	– –	✗

Table 1: KLM vs Majority Default Consequence

vs. small' distinction used in (Koutras and Rantsoudis 2017). It allows for a fine treatment of 'size' in the infinite and opens the door to analytic and number-theoretic methods. Still, the limitation to the 'countably infinite' is very restrictive and other approaches to 'size', allowing for arbitrary cardinalities, should be investigated. In this section we will explore the landscape of topological methods, identifying '*largeness*' with '*topological thickness*' and exploiting the notion of (topological) *density* which measures how '*thickly*' one set is inside another. **Topology** (also called **General Topology** or **Point-Set Topology**) is a useful, flexible and *logic-friendly* tool. Before proceeding to the precise definition of topological spaces and their properties, it is useful to review quickly the standard topology of $\mathbb{R}$. The objective is two-fold: to obtain a valuable intuition for the abstract notions of point-set topology and acquire a rich supply of examples and (in particular) counterexamples for our results in this section. The coverage is very brief and a certain familiarity of the reader is assumed; for the elements of Topology we refer the reader to (Adams and Franzosa 2007) and for the standard topology of $\mathbb{R}$ to (Abbott 2015, Ch. 3) and (Hunter 2014, Ch. 5).

The standard topology of $\mathbb{R}$. A topology on a set is completely defined by its *open* sets. To describe the so-called *standard* (or *basic*) *topology of the reals*, we have to recall some fundamental notions from analysis.

- the **open interval** $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$ and the **closed interval** $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$, (for $a, b \in \mathbb{R}$, $a < b$).
- the **half-open** (**half-closed**) **intervals** of the form $[a, b)$ and $(a, b]$, **infinite open** (or **closed**) **intervals** of the form $(a, +\infty)$, $(-\infty, b)$ ($[a, +\infty)$, $(-\infty, b]$).
- the ϵ**-neighborhood** of $a \in \mathbb{R}$, $V_\epsilon(a) = (a - \epsilon, a + \epsilon) := \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$.
- the **open sets** of the **standard topology** of $\mathbb{R}$ are those sets which can be obtained as a countable union of disjoint open intervals. The **closed sets** are the sets whose complement is open.
- given $A \subseteq \mathbb{R}$, the **closure of** A, denoted by $\mathsf{Cl}(A)$, is the *smallest closed set which contains* A and the **interior of** A, $\mathsf{Int}(A)$, is the largest open set living inside A.

Dense sets in the standard topology of $\mathbb{R}$. Density is the notion which formalises topological 'thickness' and is best described with *the* canonical example: $\mathbb{Q}$ is **dense in** $\mathbb{R}$**.** Various countably infinite sets 'live' inside $\mathbb{R}$, including $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, but the rationals have an important property not shared by the other sets: they are so densely distributed inside $\mathbb{R}$ that the reals can be 'reconstructed' out of them - and this is actually how the reals are 'constructed' via the Cauchy sequences (Moschovakis 2006, App. A). Technically, *this means that every real number can be obtained as the limit of a sequence of rational numbers*; in the topological terminology, *every real number is a limit point of* $\mathbb{Q}$ and in compact topological notation: $\mathsf{Cl}(\mathbb{Q}) = \mathbb{R}$. Equivalent -and perhaps more intuitive - ways to state the density of $\mathbb{Q}$ in $\mathbb{R}$ include:

- every open interval of $\mathbb{R}$ contains a rational number and actually, infinitely many of them.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{Q}$ arbitrarily close to x.
- $\forall x \in \mathbb{R}$, there is a rational q_n in each interval $(x, x + \frac{1}{n})$.

It should be clear that density, the topological '*largeness*' or '*thickness*', has little to do with cardinality. Subsets of $\mathbb{R}$ with the same cardinality need not be simultaneously dense and dense subsets of $\mathbb{R}$ can be vastly different in terms of cardinality:

- the set of *irrational numbers* ($\mathbb{R} \setminus \mathbb{Q}$), of cardinality $2^{\aleph_0}$, is dense in $\mathbb{R}$.
- the set $X = \{ \frac{n}{2^m} \mid n \in \mathbb{Z},\ m \in \mathbb{Z}^+\}$ is dense in $\mathbb{R}$.

The notion of density in the standard topology of $\mathbb{R}$ can be '*localized*' to subsets of the reals. Assume $A \subseteq B \subseteq \mathbb{R}$: A is called '**dense in** B' iff $\forall b \in B$ there is a sequence (a_n) of elements of A which converges to b. The set $(\mathbb{Q} \cap (0, 1))$ is not dense in $\mathbb{R}$ but it is dense in $B = (0, 1)$: $\mathsf{Cl}(\mathbb{Q} \cap (0, 1)) = [0, 1]$. Inside subsets of $\mathbb{R}$, the situation is even more complicated as the following example witnesses: let $A = \{nr + m \mid n, m \in \mathbb{Z}\}$, $r \in \mathbb{Q}$ and $B = \{na + m \mid n, m \in \mathbb{Z}\}$, $a \in (\mathbb{R} \setminus \mathbb{Q})$. It can be proved that $B \cap [0, 1]$ is dense in $[0, 1]$ while $A \cap [0, 1]$ is not.

Nowhere dense sets in $\mathbb{R}$. The real line has a very rich structure. The dense sets of reals are very important: they are the topologically 'thick' sets, sometimes called '*sets in close order*'. In the other extreme of the spectrum, the topologically '*thin*' sets are the **nowhere dense sets**. The definition conveys the intuition: a set $A \subseteq \mathbb{R}$ is called *nowhere dense in the standard topology of the reals* iff its closure contains no nonempty open sets, i.e. $\mathsf{Int}(\mathsf{Cl}(A)) = \emptyset$. The nowhere dense sets of reals are negligible in 'size' from the topological perspective and they are not dense in any non-empty subset of $\mathbb{R}$. Simple examples of nowhere dense sets abound:

- $\mathbb{N}$ and $\mathbb{Z}$, both of cardinality $\aleph_0$, are nowhere dense in $\mathbb{R}$.
- $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$.

Density nowhere in $\mathbb{R}$ is not a trivial strengthening of being '*non dense*'. Assume $X = \mathbb{Z} \cup [(0, 1) \cap \mathbb{Q}]$ which is not dense in $\mathbb{R}$ but is not nowhere dense in $\mathbb{R}$ as it is dense in $[0, 1]$: $\mathsf{Int}(\mathsf{Cl}(X)) = (0, 1)$. This may also explain why the not-nowhere-dense sets are sometimes called *somewhere dense sets*. Some useful facts about nowhere dense sets:

- every subset of a nowhere dense set is nowhere dense.
- finite unions of nowhere dense sets are nowhere dense.
- the complement of a nowhere dense set is a set with dense interior.

There is not enough space to discuss **the standard topology on the Euclidean Plane** $\mathbb{R}^2$, which is a *product topology*. For the moment, it suffices to say that the *open sets* of this topology are countable unions of *open balls of radius* ϵ *centered at* $x \in \mathbb{R}$, the closed sets are (as always) the complements of open sets and any *line* or *simple closed curve* is a nowhere dense set.

We are now ready to leave the standard topology of $\mathbb{R}$ and proceed to General Topology. A **topological space** is a pair

$\langle W, \tau \rangle$ which consists of a set W and a family τ of subsets of W satisfying

(t_1) $W, \emptyset \in \tau$

(t_2) τ is closed under finite intersections

(t_3) τ is closed under arbitrary unions

τ is called a **topology** on W, the elements of W are the **points** of the space, the members of τ are the **open sets** of the topology and the complements of open sets are the **closed sets** of τ. Sets which are both closed and open, are called **clopen sets**. The **interior operator** and the **closure operator**, $\mathsf{Int}(A)$ and $\mathsf{Cl}(A)$ respectively, are defined exactly as in the standard topology of $\mathbb{R}$. Important properties of the closure operator include:

[C1] $\mathsf{Cl}(\emptyset) = \emptyset$

[C2] $A \subseteq \mathsf{Cl}(A)$

[C3] $\mathsf{Cl}(A \cup B) = \mathsf{Cl}(A) \cup \mathsf{Cl}(B)$

[C4] $\mathsf{Cl}\Big(\mathsf{Cl}(A)\Big) = \mathsf{Cl}(A)$

[C5] $(A \subseteq B) \implies \mathsf{Cl}(A) \subseteq \mathsf{Cl}(B)$

It is known that topological spaces can be equivalently defined via a *closure operator* satisfying [C1]-[C4] ([C5], the *monotonicity of the closure operator* is a derived property) (Adams and Franzosa 2007). Similar things hold for the *interior operator*: [I1] $\mathsf{Int}(W) = W$, [I2] $\mathsf{Int}(A) \subseteq A$,[I3] $\mathsf{Int}(A \cap B) = \mathsf{Int}(A) \cap \mathsf{Int}(B)$, [I4] $\mathsf{Int}(\mathsf{Int}(A)) = \mathsf{Int}(A)$ and [I5] $(A \subseteq B) \implies \mathsf{Int}(A) \subseteq \mathsf{Int}(B)$. Exactly as in the standard topology of $\mathbb{R}$, a set X is **dense** in W iff $\mathsf{Cl}(X) = W$ and is **nowhere dense** in W iff $\mathsf{Int}(\mathsf{Cl}(X)) = \emptyset$. There exist many interesting topologies around, we will refer below to a few simple examples. The reader is referred to (Adams and Franzosa 2007) for many examples and applications.

Example 5.1 Assume a nonempty set W and let τ_d be its powerset (the collection of all subsets of W). Clearly, τ_d is a topology on W, it is called the **discrete topology** on W and this is the largest topology that can be defined on W. Every set is *clopen* in the discrete topology, the only dense set is W and the only nowhere dense set is $\emptyset$.
Assume now a collection τ_{fc} that contains only $\emptyset$ and every *cofinite* subset of W (*cofinite* is a set with a finite complement). It can be checked that τ_{fc} is a topology on W, the **finite complement topology** on W; this topology is interesting if W is infinite, in the finite case it collapses to the discrete topology. The closed sets of τ_{fc} are the finite subsets of W and W itself, the *dense sets* of this topology are the infinite sets (W is the only closed set containing an infinite set) and the *nowhere dense sets* are the finite sets.

Our **models** are of the form $M = \langle \langle W, \tau \rangle, V \rangle$ where τ is a topology on W. The following definition attempts to capture the desiderata of Section 1 with the tools of topology. Condition (ξ_1) requires that the set of $(\alpha \wedge \beta)$-worlds is '*large*' in the set of α-worlds but condition (ξ_2) asks for something much stronger than we would expect after the discusion in Section 1: it imposes that the set of $(\alpha \wedge \neg\beta)$-worlds is globally '*negligible*', not only '*small inside the set of α-worlds*'. We will comment on this below.

Definition 5.2 Assume a model $\mathcal{M} = \langle \langle W, \tau \rangle, V \rangle$ where τ is a topology on W. Then, $(\alpha \mid\!\sim \beta)$ is valid in $\mathcal{M}$ iff both conditions below are true:

(ξ_1) $\mathsf{Cl}(A \cap B) \supseteq A$ [intuitively: $(A \cap B)$ is *dense in* A]

(ξ_2) $(A \cap \overline{B})$ is nowhere dense in W

Comment 5.3 (i) It is natural to ask why we have to use (ξ_2). Condition (ξ_1) seems already strong and natural: it requires that $(A \cap B)$ is '*large*' - *topologically dense* in A. The answer is that (ξ_1) is inadequate. Indeed, we face the 'odd/even' paradoxical situation (see Section 1) in a stronger form. Let $A = \mathbb{R}$ and $B = \mathbb{Q}$. Then $(A \cap B) = \mathbb{Q}$ is dense in $\mathbb{R}$, $\mathsf{Cl}(A \cap B) = \mathsf{Cl}(\mathbb{Q}) = \mathbb{R}$ and (ξ_1) is satisfied. On the other hand, the complement of $(A \cap B)$ (wrt A), that is $(A \cap \overline{B}) = (\mathbb{R} \setminus \mathbb{Q})$ is the set of irrational numbers which is also dense in A (=$\mathbb{R}$) and moreover it is also much bigger in set-theoretic terms: its cardinality is $2^{\aleph_0} = |\mathbb{R}|$.
(ii) It seems reasonable to ask whether condition (ξ_1) is redundant in Def. 5.2 in view of the very strong (ξ_2) requirement: is $(\xi_2) \implies (\xi_1)$ true? The answer is negative. Assuming the standard topology on $\mathbb{R}$, let $A = \{1, 2, 3, 4\}$ and $B = [2, 3]$ which implies $\overline{B} = (-\infty, 2) \cup (3, +\infty)$. Then, $A \cap \overline{B} = \{1, 4\}$ is nowhere dense in $\mathbb{R}$. However, $\{2, 3\} = \mathsf{Cl}(A \cap B) = \mathsf{Cl}(\{2, 3\}) \not\supseteq A = \{1, 2, 3, 4\}$. So (ξ_1) is really needed.
(iii) We could consider a $(\xi_2)'$ of the form $\mathsf{Cl}(A \cap \overline{B}) \not\supseteq A$, which is weaker that (ξ_2) (just asking that the '*complement*' is not '*dense in* A'). However, $(A \cap \overline{B})$ could still conceivably be '*dense somewhere inside* A' and thus is not satisfactory. Def. 5.2 seems to work, in view of the results in Theorem 5.4. In particular, depending on the final status of the pending questions, it *might* prove to be more successful than we initially expected; see below.

Theorem 5.4 The consequence relations constructed with Definition 5.2 validate (and invalidate) the inferential principles denoted in Table 1.

PROOF. (**RW**) Inspecting the antecedent of the rule:

$\models (\alpha \to \beta)$ implies $A \subseteq B$ (1), hence $(C \cap A) \subseteq (C \cap B)$ and by [C5]: $\mathsf{Cl}(C \cap A) \subseteq \mathsf{Cl}(C \cap B)$ (2)

$(\gamma \mid\!\sim \alpha)$ implies

(ξ_1) $(C \cap A)$ is dense in C, i.e. $\mathsf{Cl}(C \cap A) \supseteq C$ (3)

(ξ_2) $(C \cap \overline{A})$ is nowhere dense in W (4).

To prove the consequent, we have to show that

$(\xi_1)'$ $(C \cap B)$ is dense in C, i.e. $C \subseteq \mathsf{Cl}(C \cap B)$
proof: by (3) we have that $C \subseteq \mathsf{Cl}(C \cap A)$ and by (2) $\mathsf{Cl}(C \cap A) \subseteq \mathsf{Cl}(C \cap B)$, so $C \subseteq \mathsf{Cl}(C \cap B)$ follows.

$(\xi_2)'$ $(C \cap \overline{B})$ is nowhere dense in W
proof: by (1) we obtain $\overline{B} \subseteq \overline{A}$, thus $(C \cap \overline{B}) \subseteq (C \cap \overline{A})$. The result follows from (4) and the fact that every subset of a nowhere dense set is nowhere dense.

(**CUT**)) By the antecedent of the rule:

$(\alpha \mid\!\sim \beta)$ implies

(ξ_1) $(A \cap B)$ is dense in A: $A \subseteq \mathsf{Cl}(A \cap B)$ (1)

(ξ_2) $(A \cap \overline{B})$ is nowhere dense in W (2)

$(\alpha \wedge \beta) \mid\!\sim \gamma$ implies

$(\xi_1)'$ $(A \cap B \cap C)$ is dense in $(A \cap B)$: i.e. $(A \cap B) \subseteq \mathsf{Cl}(A \cap B \cap C)$ which gives $\mathsf{Cl}(A \cap B) \subseteq \mathsf{Cl}(\mathsf{Cl}(A \cap B \cap C)) = \mathsf{Cl}(A \cap B \cap C)$ (3)

$(\xi_2)'$ $(A \cap B \cap \overline{C})$ is nowhere dense in W (4)

To prove the consequent, we have to show that

$(\xi_1)''$ $(A \cap C)$ is dense in A: $A \subseteq \mathsf{Cl}(A \cap C)$
proof: note first that $(A \cap B \cap C) \subseteq (A \cap C)$ and thus $\mathsf{Cl}(A \cap B \cap C) \subseteq \mathsf{Cl}(A \cap C)$ (5). By (1), (3) and (5) the result follows.

$(\xi_2)''$ $(A \cap \overline{C})$ is nowhere dense in W
proof: we have that $(A \cap \overline{C}) = (A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap \overline{C})$. By (4) $(A \cap B \cap \overline{C})$ is nowhere dense in W. Furthermore, $(A \cap \overline{B} \cap \overline{C}) \subseteq (A \cap \overline{B})$ which means that it is also nowhere dense in W since $(A \cap \overline{B})$ is nowhere dense in W by (2). The result follows from the fact that finite unions of nowhere dense sets are nowhere dense.

(CM) For a counterexample, let $W = \mathbb{R}^2$ endowed with the standard topology. Let $A = \mathbb{R}$, $B = \mathbb{Q}$ and $C = (\mathbb{R} \setminus \mathbb{Q})$. The antecedents of the rule are satisfied:

(ξ_1) $\mathsf{Cl}(B \cap A) = \mathsf{Cl}(\mathbb{Q}) = \mathbb{R} \supseteq A$.

(ξ_2) $(A \cap \overline{B}) = \left[\mathbb{R} \cap (\mathbb{R}^2 \setminus \mathbb{Q})\right] = C$, the irrationals, a set which is nowhere dense in $\mathbb{R}^2$.

$(\xi_1)'$ $\mathsf{Cl}(C \cap A) = \mathsf{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \supseteq A$.

$(\xi_2)'$ $(A \cap \overline{C}) = \mathbb{Q}$, the rationals, a set which is nowhere dense in $\mathbb{R}^2$.

But the consequent of the rule fails:
$(\xi_1)''$ $\mathsf{Cl}(A \cap B \cap C) = \mathsf{Cl}(\emptyset) = \emptyset \not\supseteq (A \cap B) = \mathbb{Q}$.

(Contraposition). For a counterexample, assume the *finite complement topology* on $\mathbb{N}$ (Example 5.1). Let A be a finite subset of $\mathbb{N}$ and let $B = \mathbb{N}$. We verify the antecedent $\alpha \mid\!\sim \beta$ of the rule. For (ξ_1): $(A \cap B) = A$ implies $\mathsf{Cl}(A \cap B) = \mathsf{Cl}(A) \supseteq A$ (by [C2]). For (ξ_2): $(A \cap \overline{B}) = (A \cap \emptyset) = \emptyset$ which is nowhere dense in $\mathbb{N}$ under the finite complement topology. For the consequent of the rule, notice that $\overline{A}$ is cofinite (and thus, open), $(\overline{A} \cap \overline{B}) = \emptyset$ and $\mathsf{Cl}(\overline{A} \cap \overline{B}) = \mathsf{Cl}(\emptyset) \not\supseteq \overline{A}$, hence (ξ_1) is violated. ∎

Discussion and Future Work. For a (long) moment in the timeline of this research, we hoped that we would be able to capture the '*conservative core*' of NMR via the topological incarnation of the 'most' quantifier from Def. 5.2. Although this hope has been denied, in view of the counterexample for **CM**, an inspection of Table 1 reveals that topology is interesting and promising in investigations of this kind. Technically, the proper topological context to check the 'largeness' of B inside A, is the **subspace topology on** $A \subseteq W$ (Adams and Franzosa 2007, Ch. 3.1), which is the actual content of condition (ξ_1) in Def. 5.2. It seems though that a similar notion of a set being '*nowhere dense in* $A \subseteq W$' (and not in W) has not been investigated in Point-Set Topology or even in the standard topology of $\mathbb{R}$; most probably, analysts and topologists did not need it. It is not clear to us how productive could it be to map the ('*largeness*' and '*smallness*') requirements of Section 1 entirely on the subspace topologies generated by the truth sets of our formulae and the complexity seems unmanageable at a first glance. But it is certainly a direction worth touching. A direction, parallel to this, would be to seek topological models for the KLM logics via topological duality theorems which has proven to be useful for modal logics.

In connection with the techniques of Section 4, there exist deep results relating the notion of *asymptotic density of integer sequences* with the notion of *topological density*. Assume $X \subseteq \mathbb{N}^*$ and let $X_f = \{m/n \mid m, n \in X\}$. If X has positive asymptotic density then X_f is dense in $\mathbb{R}^+$. If X_f is not dense in $\mathbb{R}^+$ then $\mathsf{d}_*(X) < 1/2$ and $\mathsf{d}^*(X) < 1$ (Mišik and Tóth 2003). It is clear that the analytic and topological notions we have used are somehow related and it remains to check whether they can be combined to provide a better capture of 'majority'.

6 Default Consequence via Measure

Measure theory has been developed within *Mathematical Analysis* with an aim of improving the notion of integral and it is intimately related to *Probability Theory*. A *measure* on a set X is a well-defined systematic way to assign a real number - intuitively conceived as its '*size*' - to any '*admissible*' subset of X. This number can be alternatively conceived as the '*mass*' of the subset, assuming a mass distribution throughout the space. A measure function generalizes *length* in $\mathbb{R}$, *area* in $\mathbb{R}^2$, *volume* in $\mathbb{R}^3$ to obtain *mass* in the Euclidean Space $\mathbb{R}^n$. We refer the reader to (Folland 1999, Ch. 1) for more on *Measure Theory*.

The families of sets which can serve as domains of measure functions are the σ-algebras. A σ-algebra on W is a non-empty collection of subsets of W, closed under *countable unions* and *complements*. It can be easily checked that a σ-algebra is also *closed under countable intersections* and thus always includes $\emptyset$ and W. To keep matters simple, and since our sets will actually be sets of the form $\|\beta\|_{\mathcal{M}}$ for a formula β, **we will assume only powerset algebras** which are, trivially, σ-algebras.

Assume a set W and a σ-algebra M on W. A **measure on** (W, M) is a function $\mu : M \to [0, \infty]$ (*non-negativity*) s.t.:

(m_1) $\mu(\emptyset) = 0$

(m_2) for E_i, $i \in \mathbb{N}$, a sequence of disjoint sets in M

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \text{ (\textit{countable additivity})}$$

The triple $\langle W, M, \mu \rangle$ is called a **measure space**. The measure function is **monotonic**: $(A \subseteq B) \Rightarrow (\mu(A) \leq \mu(B))$. It is worth noting here that **measure captures completely different aspects of size than topological density and/or set cardinality**. Moreover, the situation can be surprising and extremely complicated too. Again, the 'canonical' example is $\mathbb{Q}$ but other examples abound.

> $\mathbb{Q}$ can be '*large*' or (very) '*small*', depending on the perspective: "The count of individual rational numbers is *huge* – there are *infinitely many*. Rationals are *dense* in the reals – they are *everywhere*. However, if we consider the portion of the real line formed by the rationals, it is significantly *insignificant* – the rationals have *measure zero*. So $\mathbb{Q}$ is both *very large* and *very small*." (Bauldry 2009).

the **Cantor set** C is an ingenious construction (Abbott 2015, Ch. 3.1). It is obtained by removing the open middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$ and repeating this process for every emerging interval. Alternatively, it can be defined as the set of all $x \in [0, 1]$ that have a base-3 expansion $\sum a_j 3^{-j}$, $a_j \neq 1$ for all j (Folland 1999, p. 38). The Cantor set is set-theoretically *huge* - $|C| = 2^{\aleph_0} = |\mathbb{R}|$, but it is nowhere dense in $\mathbb{R}$ - hence topologically negligible, and measure-theoretically 'tiny' as $\mu(C) = 0$.

the rationals have measure *zero* but, surprisingly, the set of irrationals in $[0, 1]$ have measure *one*, even though they have the same cardinality with the Cantor Set.

Within a measure space $\langle W, M, \mu\rangle$, a set $A \in M$ such that $\mu(A) = 0$ is called a **null set** and is considered measure-theoretically '*small*'. A statement about points of W is said to be '**true almost everywhere**' if it is true in W except the points within some *null set*. Noticing that our models in this section are of the form $\mathcal{M} = \langle\langle W, \mu\rangle, V\rangle$ where the measure function μ is defined on the *powerset algebra* of W, we immediately understand that measure seems perfectly tailored for majority-default reasoning. Following the intuition described in Section 1, we would like to model $(\alpha \mid\!\sim \beta)$ by asking that $\mu(A \cap B) = \mu(A)$ ($(A \cap B)$ is '*large*' in A) and $\mu(A \cap \overline{B}) = 0$ ($(A \setminus B)$ is a measure-theoretically '*null*' set). A first observation is that these conditions are equivalent.

Fact 6.1 $\mu(A \cap B) = \mu(A) \;\Leftrightarrow\; \mu(A \cap \overline{B}) = 0$

PROOF. Countable additivity confirms that

$$\mu(A) = \mu(A \cap B) + \mu(A \cap \overline{B})$$

Both directions follow immediately. ∎

It is now tempting to expect that we could have two birds with one shot. Unfortunately, it seems that this is a very strong condition.

Fact 6.2 Set $(\alpha \mid\!\sim \beta) \;\Leftrightarrow\; \mu(A \cap \overline{B}) = 0$. The emerging consequence relation is monotonic.

PROOF. By the antecedent of the rule: $\mu(A \cap \overline{B}) = 0$. We observe that $(A \cap C \cap \overline{B}) \subseteq (A \cap \overline{B})$ which implies (by *monotonicity of measure*) $\mu(A \cap C \cap \overline{B}) \leq \mu(A \cap \overline{B})$. So, $\mu(A \cap C \cap \overline{B}) = 0$ and the consequent of the rule $(\alpha \wedge \gamma) \mid\!\sim \beta$ is true. ∎

We finally proceed to the following definition, actually the measure-theoretic equivalent of *simple majority*.

Definition 6.3 Let $\mathcal{M} = \langle\langle W, \mu\rangle, V\rangle$ be a model based on the measure space $\langle W, 2^W, \mu\rangle$. Then $\alpha \mid\!\sim \beta$ is valid in M iff either (λ_1) $\mu(A) = 0$ or:

$$(\lambda_2) \quad \mu(A \cap B) > \frac{1}{2}\mu(A)$$

Condition (λ_1) corresponds to a 'vacuously' satisfied conditional and is only required for axiom **REF**. The following Theorem collects the properties of the default system introduced with Def. 6.3.

Theorem 6.4 The consequence relations constructed with Definition 6.3 validate (and invalidate) the inferential principles denoted in Table 1.

PROOF. **(D).** (*Case* 1). Assume that $(\alpha \wedge \beta) \mid\!\sim \gamma$ is fired by (λ_1). This means that $\mu(A \cap B) = 0 \;\Leftrightarrow\; \mu(A \cap \overline{B}) = \mu(A)$ (1) (Fact 6.1). By the *monotonicity of measure*: also $\mu(A \cap B \cap C) = 0$ (2). If the other conditional of the antecedent $(\alpha \wedge \neg\beta) \mid\!\sim \gamma$ is also fired by (λ_1), then (by (m_2)) $\mu(A) = \mu(A \cap B) + \mu(A \cap \overline{B}) = 0$ and we are done. Otherwise, it is fired by (λ_2) and thus $\mu(A \cap \overline{B} \cap C) > \frac{1}{2}\mu(A \cap \overline{B})$ (3). Then $\mu(A \cap C) \overset{(m_2)}{=} \mu(A \cap B \cap C) + \mu(A \cap \overline{B} \cap C)$ $\overset{(2)}{=} 0 + \mu(A \cap \overline{B} \cap C) \overset{(3)}{>} \frac{1}{2}\mu(A \cap \overline{B}) \overset{(1)}{=} \frac{1}{2}\mu(A)$ and we are done. The case for the other element of the antecedent is completely symmetric.

(*Case* 2). Assume the rule is fired by (λ_2). Then $\mu(A \cap C) = \mu(A \cap C \cap (B \cup \overline{B})) = \mu\Big([A \cap B \cap C] \cup [A \cap \overline{B} \cap C]\Big) \overset{(m_2)}{=}$ $\mu(A \cap B \cap C) + \mu(A \cap \overline{B} \cap C) \overset{(\lambda_2)}{>} \frac{1}{2}\mu(A \cap B) + \frac{1}{2}\mu(A \cap \overline{B})$ $= \frac{1}{2}\Big[\mu(A \cap B) + \mu(A \cap \overline{B})\Big] \overset{(m_2)}{=} \frac{1}{2}\mu(A)$. ∎

Discussion and Future Work. The emerging logics from our measure-theoretic investigations on 'majority'-default consequence are *supraclassical*, *nonmonotonic* and they satisfy the ubiquitous axiom **REF**. They also validate the rules **LLE** (*Left Logical Equivalence*) and **RW** (*Right Weakening*) which are fundamental, in the sense that it is difficult to imagine a reasonable commonsense reasoning mechanism that would not validate them. So, in a very precise sense, they capture a set of minimum requirements for default reasoning and they come very close to probabilistic consequence (Delgrande and Renne 2016); this is true to a large extent also for the consequence relation defined in Section 4. Measure theory offers more advanced perspectives for measuring the 'size' of $\mathbb{R}$ and actually, the size of any complete metric space. This area develops around the famous *Baire Category Theorem* (Abbott 2015, p. 109). We know from Topology that the *nowhere-dense sets* are the topologically '*thin*' sets. Any set which can be formed by a countable union of nowhere dense sets, is called a '*meager*' set or a set of '*first category*'. A set that is not of first category is of 'second category' and the sets of the *second category* are the '*large*' ('*fat*') subsets. This is an advanced perspective which is worth exploring.

7 Conclusion

In this paper, we have attempted to develop *model-theoretic non-monotonic logics* by studying 'majority'-default conditionals with the use of techniques from Topology and Mathematical Analysis. We strongly believe that this is only the beginning in this avenue of research. The programme of developing majority-default conditionals based on elegant parts of classical mathematics which formalize aspects of 'size' might furnish a new perspective on default reasoning.

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