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Generalized Ranking Kinematics for Iterated Belief Revision

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Abstract

Probability kinematics is a leading paradigm in probabilistic belief change. It is based on the idea that conditional beliefs should be independent from changes of their antecedents' probabilities. In this paper, we propose a re-interpretation of this paradigm for Spohn's ranking functions which we call Generalized Ranking Kinematics as a new principle for iterated belief revision of ranking functions by sets of conditional beliefs. This general setting also covers iterated revision by propositional beliefs. We then present c-revisions as belief change methodology that satisfies Generalized Ranking Kinematics.

1 Introduction

In multi-agent systems, it is crucial for agents working together to process information from different contexts independently. Each agent collects evidence from a different part of the world, which leads to new information about the specific subcontexts. Jeffrey introduced in 1965 (Jeffrey 1965) a probabilistic revision method called *Jeffrey's Rule* which is able to deal with information coming from contexts that complement each other. Jeffrey's Rule is a generalization of Bayesian conditionalization. The key to revising probabilistic beliefs with new evidence from different subcontexts is *Probability Kinematics*, which is an important issue in probabilisitic belief revision in general and especially for Jeffrey's rule. Briefly it says that conditional probabilities should not change if the probabilities of the conditions change. Spohn (Spohn 2014) generalized the notion of Jeffrey's Rule to ordinal conditional functions (OCFs). He introduces two interchangeable forms of conditionalization based on evidence or new information, where the former one leads to a strengthening of beliefs and the latter one displays a quantified belief revision.

Shore and Johnson (Shore and Johnson 1980) introduced *Subset Independence* as a crucial property of inductive inference in a probabilistic framework. *Subset Independence* is a generalization of Probability Kinematics using a set of conditionals with antecedents implying exclusive and exhaustive cases. In this paper we present a property for revision by conditionals in the framework of ranking functions which we call *Generalized Ranking Kinematics* (GRK) that can be seen as an analog to Subset Independence for qualitative belief revision. This new property of *Generalized Ranking Kinematics* that we introduce in this paper connects Jeffrey's Rule to the axioms of inductive inference introduced by Shore and Johnson and can be seen as an extension of Spohn's work on transferring Jeffrey's rule to the framework of ranking functions. Our main contributions are as follows:

- We transfer the notion of Subset Independence to the framework of ranking functions by introducing a strong and a weak version of *Generalized Ranking Kinematics* applying to revision by sets of conditionals and allowing for reducing revisions to local contexts.
- We present an algorithm to compute the *finest splitting* of contexts for an arbitrary finite set of conditionals.
- We prove that c-revisions satisfy *Generalized Ranking Kinematics*.

The rest of the paper is organized as follows. In Section 2 we present relevant formal preliminaries. Section 3 discusses aspects of probabilistic belief revision and related work. *Generalized Ranking Kinematics* is introduced in Section 4 in the framework of ranking functions. In Section 5, we present c-revisions as a concrete example of a revision method which fulfills *Generalized Ranking Kinematics*. Finally, Section 6 contains conclusions.

2 Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet Σ with atoms $a, b, c, \ldots$ and with formulas $A, B, C, \ldots$. For conciseness of notation, we will omit the logical and-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. We call formulas A_i $(i = 1, \ldots, n)$ *exclusive* iff $A_i A_j \equiv \perp$ for $i \neq j$, and *exhaustive* iff $A_1 \vee \ldots \vee A_n \equiv \top$. The set of all propositional interpretations over Σ is denoted by Ω_Σ. As the signature will be fixed throughout the paper, we will usually omit the subscript and simply write Ω. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $Mod(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $Mod(A) \subseteq Mod(B)$, as usual. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

$\mathcal{L}$ is extended to a conditional language $(\mathcal{L}|\mathcal{L})$ by introducing a conditional operator $|$: $(\mathcal{L}|\mathcal{L}) = \{(B|A)|A, B \in \mathcal{L}\}$. $(\mathcal{L}|\mathcal{L})$ is a flat conditional language, no nesting of conditionals is allowed. Conditionals are usually considered within richer semantic structures such as *epistemic states*. We briefly describe a well-known representation of epistemic states: *Ordinal Conditional Functions* (OCF, also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$ express degrees of plausibility and were firstly introduced by Spohn (Spohn 1988). We have $\kappa(A) := \min\{\kappa(\omega)|\omega \models A\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. A proposition A is believed if $\kappa(\overline{A}) > 0$ and conditionals are assigned a degree of plausibility by setting $\kappa(B|A) = \kappa(AB) - \kappa(A)$. Conditionals are accepted in an epistemic state represented by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$. An OCF can be conditionalized by a proposition A through $\kappa(\omega|A) = \kappa(\omega) - \kappa(A)$ for $\omega \models A$. $\kappa(\cdot|A)$ is an OCF on $Mod(A)$. Instead of writing $\kappa(\cdot|A)$ we will use the shorter notation $\kappa_{|A}$. The conditionalized OCF $\kappa_{|A}$ is only defined on the set $Mod(A)$ and not for the worlds $\omega \notin Mod(A)$. OCF's can be considered as a qualitative counterpart of probability distributions. For any set M, $M = M_1 \dot{\cup} M_2$ means that M is a union of disjoint sets M_1, M_2.

3 Probability Kinematics in Belief Revision and Related Work

In a probabilistic framework, Bayesian conditionalization is a well-known method to guarantee the success postulate for the revision of an epistemic state with new information A. We obtain the posterior probability distribution by conditionalizing the prior with A (for $P(A) > 0$):

$$P * A(\omega) = P(\omega|A) = \begin{cases} \frac{P(\omega)}{P(A)}, & \omega \models A \\ 0, & \omega \models \overline{A} \end{cases} \text{ for } \omega \in \Omega. \quad (1)$$

Jeffrey introduced a generalization of the Bayesian conditionalization in 1965 (Jeffrey 1965), where the new information consists of assigning new probabilities $P^*(A_i) = y_i$ to a set of exclusive and exhaustive formulas $A_1, \ldots, A_n$ so that $\sum_{i=1}^{n} P^*(A_i) = 1$. Jeffrey's rule is based on a strong assumption which is called *Probability Kinematics*: The new information $P^*(A_i) = y_i$ does not change the conditional probability given A_i:

$$P^*(B|A_i) = P(B|A_i) \quad (2)$$

Jeffrey's rule results from the law of total probability:

$$P^*(B) = \sum_{i=1}^{n} P(B|A_i) P^*(A_i). \quad (3)$$

Jeffrey's rule displays a multiple probabilistic revision method for a set of probabilistic facts $\mathcal{S} = \{A_1[x_1], \ldots, A_n[x_n]\}$ such that $P * \mathcal{S} \models \mathcal{S}$. Here, a probability distribution P satisfies a probabilistic fact $A_i[x_i]$, $P \models A_i[x_i]$ iff $P(A_i) = x_i$. Similiarly, for a probabilistic conditional $(B_i|A_i)[x_i]$, $P \models (B_i|A_i)[x_i]$ iff $P(B_i|A_i) = x_i$. There have been several proposals to generalize Jeffrey's rule. For example, Wagner (cf. (Wagner 1992)) uses an arbitrary set of propositions A_i. Smets generalized Jeffrey's rule to belief functions (see (Smets 2013)), and Benferhat et al. analyse the expressive power of possibilistic counterparts to Jeffrey's rule for modeling belief revision (Benferhat et al. 2010).

Shore and Johnson (Shore and Johnson 1980) proposed a far-reaching generalization of Probability Kinematics under the name *Subset Independence*:

Definition 1 (Subset Independence for Probability Distributions)**.** *Let $A_1, \ldots, A_n$ be exhaustive and exclusive formulas. Let P be a probability distribution and $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n$ be a set of probabilistic conditionals with subsets $\mathcal{R}_i$ whose premises imply A_i, and $\mathcal{S} = \{A_1[x_1], \ldots, A_n[x_n]\}$ with $\sum_{i=1}^{n} x_i = 1$. The revision operator $*$ satisfies* Subset Independence *iff*

$$(P * (\mathcal{R} \cup \mathcal{S}))(\cdot|A_i) = P(\cdot|A_i) * \mathcal{R}_i. \quad (4)$$

For explanation and motivations see (Shore and Johnson 1980). The Probability Kinematics assumption that is crucial for Jeffrey's rule follows immediately from Subset Independence if we take the new information $\mathcal{R}$ to be the empty set. Let $*$ be a revision operator satisfying Subset Independence then $(P * \mathcal{S})(\cdot|A_i) = P(\cdot|A_i)$, so for every $B \in \mathcal{L}$ we have $(P * \mathcal{S})(B|A_i) = P(B|A_i)$ which is (2). In the next section we want to close the gap between the generalization of Probability Kinematics in a quantitative framework and the qualititative version of Jeffrey's rule by transferring the notion of Subset Independence to the OCF framework. Note that our framework goes far beyond the classical AGM belief revision theory (Alchourrón, Gärdenfors, and Makinson 1985) and the approach of iterated revision by Darwiche and Pearl (Darwiche and Pearl 1997) because it deals with revision by sets of conditionals, but is fully compatible with these seminal frameworks.

4 Generalized Ranking Kinematics for OCFs

In this section we will transfer the idea of Probability Kinematics (2) for probability distributions to the framework of ranking functions by re-interpreting the property of Subset Independence for the OCF framework. We distinguish between a strong and a weak version here.

Definition 2 (Generalized Ranking Kinematics (GRK) for OCF)**.** *Let $A_1, \ldots, A_n$ be exhaustive and exclusive formulas. Let κ be an ordinal conditional function and $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n$ be a set of conditionals, with subsets $\mathcal{R}_i$ whose premises imply A_i, and $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \ldots, n\}$. The revision operator $*$ satisfies* strong Generalized Ranking Kinematics *iff*

$$(GRK^{strong}) \quad \kappa * (\mathcal{R} \cup \{S\})(\cdot|A_i) = \kappa(\cdot|A_i) * \mathcal{R}_i. \quad (5)$$

We can also define a weak Generalized Ranking Kinematics *for OCFs:*

$$(GRK^{weak}) \qquad \kappa * \mathcal{R}(\cdot|A_i) = \kappa(\cdot|A_i) * \mathcal{R}_i. \quad (6)$$

Generalized Ranking Kinematics expresses two strong irrelevance assertions: Firstly, if we condition the revised OCF by a case A_i, then only the conditionals talking about this case are relevant. Secondly, the plausibility of a case A_i is not relevant for revising the respectively conditionalized OCF. In the probabilistic case S is not a formula but the set which assigns posterior probabilities to the formulas A_i, for OCFs we just take into account that some of the A_i's are more plausible than others and revise with the disjunction of them. Note that exclusivity is the crucial property of the A_i's because exhaustiveness can always be obtained by taking the negation of $\bigvee A_i$ also as a case.

We give an example to illustrate *Generalized Ranking Kinematics*:

Example 1. *An agent has a new friend and they make plans for the weekend, but the new friend is not sure yet if he is really free on Saturday. So, if the friend has to work during the weekend the agent will go to the gym alone. If the friend does not have to work and the weather is sunny, they will go to the park, else they will go to the cinema together. We formalize this situation: Let $\Sigma = \{w, g, s, p, c\}$ be the signature, where:*

w: *the agent's friend has to work;*
g: *the agent goes to the gym;*
s: *the weather is sunny;*
p: *the agents go to the park;*
c: *the agents go to the cinema.*

The weekend plans of the agent depend on whether the friend has to work or not, which means we have (at least) two exclusive scenarios which we can revise independently.

In the following we will investigate the preconditions of the strong and weak version of *Generalized Ranking Kinematics* in more detail. We shall at first explain the notion of *premise splitting* which is the main precondition for Subset Independence: The premises must imply formulas A_i which are exclusive and exhaustive. This means the agent is able to distinguish between different cases. During the revision the agent learns something about the consequences resulting from these cases. The definition of premise splitting is as follows:

Definition 3 (Premise splitting)**.** *Let $\mathcal{R} = \{(B_1|X_1), \ldots, (B_m|X_m)\}$ be a set of conditionals. A* premise splitting $\mathcal{P}_\mathcal{R}$ of $\mathcal{R}$ *is a set $\mathcal{P}_\mathcal{R} = \{A_1, \ldots, A_n\}$ of exclusive and exhaustive formulas $A_1, \ldots A_n$, such that every premise X_k ($k = 1, \ldots, m$) implies exactly one A_i.*

We continue example 1 to explain the notion of premise splitting:

Example 2. *The following set of conditionals models the agent's weekend plans: $\mathcal{R} = \{(g|w), (p|\overline{w}s), (c|\overline{w}\,\overline{s})\}$. Every premise in $\mathcal{R}$ is implied by one of the exclusive and exhaustive formulas $A_1 = w$, $A_2 = \overline{w}s$ and $A_3 = \overline{w}\,\overline{s}$, which leads us to the following premise splitting $P_\mathcal{R} = \{w, \overline{w}s, \overline{w}\,\overline{s}\}$. This illustrates how sets of conditionals that fulfill the conditions of* Generalized Ranking Kinematics *model exclusive and exhaustive cases, meaning that they describe the agent's actions for every possible scenario.*

A premise splitting induces a partitioning of the set of conditonals $\mathcal{R} = \{(B_1|X_1), \ldots, (B_m|X_m)\}$. We can define subsets $\mathcal{R}_i$ of $\mathcal{R}$ by: $(B|X) \in \mathcal{R}_i$ iff $X \vDash A_i$. The subsets are disjoint since the A_i's are exclusive, i.e. $\mathcal{R} = \mathcal{R}_1 \dot\cup \cdots \dot\cup \mathcal{R}_n$.

We can find a premise splitting for each set of conditionals $\mathcal{R}$ by using simply the trivial premise splitting $\mathcal{P}_\mathcal{R} = \{\top\}$. But in order to maximise the benefits of Subset Independence resp. *Generalized Ranking Kinematics*, the premise splitting should be as fine as possible, meaning that the subsets $\mathcal{R}_i$ should be as small as possible and as many as possible.

Definition 4 (Refinement and specificity)**.** *Let $\mathcal{R}$ be a set of conditionals. For two premise splittings $\mathcal{P}^1_\mathcal{R} = \{A_1, \ldots, A_n\}$ and $\mathcal{P}^2_\mathcal{R} = \{B_1, \ldots, B_{n'}\}$ of $\mathcal{R}$, we say that $\mathcal{P}^1_\mathcal{R}$ is a* refinement *of $\mathcal{P}^2_\mathcal{R}$ iff every B_j is implied by some A_i:*

$$\mathcal{P}^1_\mathcal{R} \leqslant \mathcal{P}^2_\mathcal{R} \text{ iff } \forall\, B_j \in \mathcal{P}^2_\mathcal{R},\ \exists\, A_i \in \mathcal{P}^1_\mathcal{R} \text{ s.t. } A_i \vDash B_j.$$

This means that the A_i's are more specific *than the B_j's.*

Two premise splittings $\mathcal{P}^1_\mathcal{R}$ and $\mathcal{P}^1_\mathcal{R}$ are equivalent iff $\mathcal{P}^1_\mathcal{R} \leqslant \mathcal{P}^2_\mathcal{R}$ and $\mathcal{P}^2_\mathcal{R} \leqslant \mathcal{P}^1_\mathcal{R}$

We continue Example 1 to illustrate the refinement-relation defined above:

Example 3. *For $\mathcal{R} = \{(g|w), (p|\overline{w}s), (c|\overline{w}\,\overline{s})\}$ (see Example 1), we could also use $B_1 = w$ and $B_2 = \overline{w}$ as exhaustive and exclusive formulas to define a premise splitting $P'_\mathcal{R} = \{w, \overline{w}\}$. Then $P_\mathcal{R} \leqslant P'_\mathcal{R}$, since $w \vDash w$ and $\overline{w}s \vDash \overline{w}$.*

A premise splitting which refines every other splitting for a set of conditionals $\mathcal{R}$ is called its *finest premise splitting*. The following theorem shows that for every $\mathcal{R}$ the finest premise splitting is unique up to semantic equivalences and permutations. To prove the theorem, we need to define a relation on the set of premises of $\mathcal{R}$, we denote this set by $Prem(\mathcal{R}) = \{X \in \mathcal{L} | (B|X) \in \mathcal{R}\}$. We start by defining $X \sim Y$ iff $XY \not\equiv \bot$ for $X, Y \in Prem(\mathcal{R})$. $\sim$ is reflexive and symmetric, so the transitive closure $\sim^*$ of $\sim$ is an equivalence relation on the elements of $Prem(\mathcal{R})$.

Theorem 1. *For a set of conditionals $\mathcal{R}$ there is a (unique) finest premise splitting (up to semantic equivalences and permutations).*

Proof. Let $Prem(\mathcal{R}) = \{X \in \mathcal{L} | (B|X) \in \mathcal{R}\}$, and $\sim^*$ be the equivalence relation defined above. Then $Prem(\mathcal{R}) = \bigcup_{i=1,\ldots,n}[X_i]$, where $[X_i]$ are the equivalence classes of $\sim^*$. Let $A_i = \bigvee[X_i]$ for $i = 1, \ldots, n$ and $A_0 = \neg(A_1 \vee \ldots \vee A_n) \equiv \overline{A_1} \ldots \overline{A_n}$, then $\mathcal{P}_\mathcal{R} = \{A_0, A_1, \ldots, A_n\}$ defines a premise splitting because for $i \neq j$ it holds that:

$$A_i A_j \equiv \Big(\bigvee_{\tilde{X} \in [X_i], \tilde{Y} \in [X_j]} (\tilde{X}\tilde{Y}) \Big) \equiv \bot,$$

and $A_0 \vee A_1 \vee \ldots \vee A_n \equiv (\overline{A_1} \ldots \overline{A_n}) \vee A_1 \ldots \vee A_n \equiv \top$. It is clear that for every premise $X \in Prem(\mathcal{R}) = \bigcup_{i=1,\ldots,n}[X_i]$, there is one $A_i = \bigvee[X_i]$ which is implied by X, and that $\mathcal{P}_\mathcal{R}$ is unique up to permutation and semantic equivalences. But we still need to show that $\mathcal{P}_\mathcal{R}$ refines every other premise splitting: Let $\mathcal{P}'_\mathcal{R} = \{B_1, \ldots, B_{n'}\}$ be another premise splitting of $\mathcal{R}$. For

$X, Y \in Prem(\mathcal{R})$ it holds that, if $X \vDash B_i$ and $Y \vDash B_j$ with $i \neq j$, then $XY \equiv \bot$. This means that for $X, Y \in Prem(\mathcal{R})$ with $XY \not\equiv \bot$, there is $i \in \{1, \ldots, n'\}$ with $X \vDash B_i$ and $Y \vDash B_i$, which means $[X] \vDash B_i$. Hence, $\exists j \in 1, \ldots, n$ with $\bigvee_{\tilde{X} \in [X_j]} \tilde{X} = A_j \vDash B_i$ and $\mathcal{P}_\mathcal{R}$ refines $\mathcal{P}'_\mathcal{R}$. $\square$

To compute the finest premise splitting for a finite arbitrary set of conditionals $\mathcal{R}$ we present Algorithm 1.

Algorithm 1 Finest premise splitting

Input: Finite set of conditionals $\mathcal{R}$
Output: Unique finest premise splitting of $\mathcal{R}$

```
 1: Prem ← Prem(R)
 2: P_R = ∅
 3: while Prem ≠ ∅ do
 4:     Choose X ∈ Prem
 5:     if there are Y ∈ Prem, Y ≠ X, with XY ≢ ⊥ then
 6:         build [X] = {Y ∈ Prem | XY ≢ ⊥}
 7:         A ← ⋁[X]
 8:         Prem ← (Prem \ [X]) ∪ {A}
 9:     else
10:         A ← X
11:         P_R ← P_R ∪ {A}
12:         Prem ← Prem \ {A}
13:     end if
14: end while
15: if ⋁ P_R ≢ ⊤ then
16:     A_0 = ⋀_{A_i ∈ P_R} ¬A_i
17:     P_R = P_R ∪ {A_0}
18: end if
19: return P_R
```

Theorem 2. *Algorithm 1 terminates and is correct in the sense that it computes the unique finest premise splitting for a finite set of conditionals $\mathcal{R}$.*

This theorem follows immediately from the constructive proof of Theorem 1 by observing that the transitive closure of $\sim$ is obtained by considering disjunctions in line 7 and adding these to the set of premises $Prem$ in line 8. For premises X_1, X_2, X_3 with $X_1X_2 \not\equiv \bot$ and $X_1X_3 \not\equiv \bot$, it might be the case that $X_2X_3 \equiv \bot$, but $(X_1 \vee X_2)X_3 \not\equiv \bot$. The running time of Algorithm 1 is determined by the SAT-Test in line 5 and 6. In the worst case the equivalence classes determined in the while-loop are singletons and we obtain $\mathcal{O}(s^2)$, where s represents the runtime of the SAT-Test.

The following example illustrates the algorithm:

Example 4. *Let $\Sigma = \{a, b, c, d\}$ be the signature for a set of conditionals $\mathcal{R} = \{(c|ab), (d|a\overline{b}c), (e|a\overline{b}), (b|\overline{a}), (d|\overline{a}e)\}$. We now want to compute the unique finest premise splitting using the Algorithm 1: We initialize $Prem = \{ab, a\overline{b}c, a\overline{b}, \overline{a}, \overline{a}e\}$ and $\mathcal{P}_\mathcal{R} = \emptyset$. In the first iteration, $X = ab$ with $A_1 = ab$ and therefore, $\mathcal{P}_\mathcal{R} = \{ab\}$ and $Prem = \{a\overline{b}c, a\overline{b}, \overline{a}, \overline{a}e\}$. For the second iteration, $X = a\overline{b}c$ with $[X] = \{a\overline{b}c, a\overline{b}\}$, hence $A_2 = a\overline{b}$ and $Prem = \{a\overline{b}, \overline{a}, \overline{a}e\}$. In the next iteration, $X = a\overline{b}$ and $XY \equiv \bot$ for all other $Y \in Prem$, therefore $\mathcal{P}_\mathcal{R} = \{ab, a\overline{b}\}$. Then, $X = \overline{a}$ with $[X] = \{\overline{a}, \overline{a}e\}$ and $A_3 = \overline{a}$ and therefore, $Prem = \{\overline{a}\}$. In the last iteration, $X = \overline{a} = A_3$, and $\mathcal{P}_\mathcal{R} = \{ab, a\overline{b}, \overline{a}\}$. We have $ab \vee a\overline{b} \vee \overline{a} \equiv \top$, thus the algorithm terminates and returns $\mathcal{P}_\mathcal{R} = \{ab, a\overline{b}, \overline{a}\}$ which determines a partitioning of $\mathcal{R}$. For the GRK property we split $\mathcal{R} = \{(c|ab)\}\dot{\cup}\{(d|a\overline{b}c), (e|a\overline{b})\}\dot{\cup}\{(b|\overline{a}), (d|\overline{a}e)\} = \mathcal{R}_1\dot{\cup}\mathcal{R}_2\dot{\cup}\mathcal{R}_3$.*

In this section we have shown that for every arbitrary set of conditionals $\mathcal{R}$, we can find a unique finest premise splitting which fulfills the preconditions of GRK. In the next section we will give a concrete example for a revision method which fulfills both strong and weak versions of *Generalized Ranking Kinematics* within the framework of ordinal conditional functions.

5 GRK for C-Revisions

In a nutshell, *Generalized Ranking Kinematics* means that if the new information that the agents receive can be split into different cases, then it should be possible to revise with these different subsets independently on the conditionalized prior epistemic state. In a purely quantitative framework, the principle of maximum entropy is a revision method which fulfills this property for probability distributions. In this section, we will focus on the OCF framework and use c-revisions introduced by Kern-Isberner in 2001 (Kern-Isberner 2001) as a proof of concept to illustrate how GRK can be implemented for iterated belief revision.

c-revisions provide a highly general framework for revising OCFs by sets of conditionals. For our purposes it will be sufficient to use a simplified version of c-revisions.

Definition 5 (c-revisions for OCFs)**.** *Let κ be an OCF specifying a prior epistemic state, and let $\mathcal{R} = \{(B_1|X_1), \ldots, (B_m|X_m)\}$ be the set of conditionals which represent the new information. Then a* c-revision of κ by $\mathcal{R}$ *is given by an OCF of the form*

$$\kappa * \mathcal{R}(\omega) = \kappa^*(\omega) = \kappa_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant m \\ \omega \vDash X_i \overline{B}_i}} \nu_i^- \tag{7}$$

with non-negative integers ν_i satisfying

$$\begin{aligned} \nu_i^- > & \min_{\omega \vDash X_i B_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \vDash X_j \overline{B}_j}} \nu_j^- \Big\} \\ & - \min_{\omega \vDash X_i \overline{B}_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \vDash X_j \overline{B}_j}} \nu_j^- \Big\}. \end{aligned} \tag{8}$$

The vector $(\nu_1^-, \ldots, \nu_m^-)$ characterizes (defines) each c-revision κ^* of κ by $\mathcal{R}$.

The ν_i^- can be considered as impact factors of the single conditionals $(B_i|X_i)$ for falsifying the conditionals in $\mathcal{R}$ which have to be chosen so as to ensure success by (8). κ_0 in (7) is a normalization factor. In the following lemma, we will further characterize κ_0, which will help us to understand the proof of Theorem 3.

Lemma 1. *Let $\mathcal{R} = \{(B_1|X_1), \ldots, (B_m|X_m)\}$, and let $\kappa * \mathcal{R} = \kappa^*$ be a c-revision of κ by $\mathcal{R}$ satisfying (7) and (8). Then $\kappa_0 = -\min_{\omega \in \Omega}\{\kappa(\omega) + \sum_{1 \leqslant i \leqslant m,\, \omega \vDash X_i \overline{B}_i} \nu_i^-\}$.*

This follows immediately from $\kappa^*(\top) = 0$. Since (7) and (8) provide a general schema for revision operators, many c-revisions are possible. Nevertheless, it might be useful to impose further constraints on the parameters ν_i^-. One option is to take minimal ν_i^- satisfying (8) ensuring that the resulting OCF ranks worlds as plausible as possible.

Definition 6 (Minimal c-revisions (Kern-Isberner and Huvermann 2017))**.** *A minimal c-revision of κ by $\mathcal{R}$ is a c-revision κ^* defined by $(\nu_1^-, \ldots, \nu_m^-)$ such that the vector $(\nu_1^-, \ldots, \nu_m^-)$ is Pareto-minimal, i.e., no other c-revision of κ by $\mathcal{R}$ which is characterized by $(\tilde{\nu}_1^-, \ldots, \tilde{\nu}_m^-)$ exists with $\tilde{\nu}_i^- \leqslant \nu_i^-$, for all i, and $\tilde{\nu}_i^- < \nu_i^-$ for at least one i, $1 \leqslant i \leqslant m$.*

Each c-revision is an iterated revision in the sense of Darwiche and Pearl (Darwiche and Pearl 1997), and it also satisfies the *Generalized Ranking Kinematics*:

Theorem 3. *Let $\mathcal{R} = \mathcal{R}_1 \dot\cup \ldots \dot\cup \mathcal{R}_n$ be a set of conditionals, with subsets $\mathcal{R}_i = \{(B_{j,i}|A_iC_{j,i})\}_{j=1,\ldots,n_i}$ $(n_i = |\mathcal{R}_i|)$ for $i = 1, \ldots, n$ such that $\{A_1, \ldots, A_n\}$ is a premise splitting. Let $S = \bigvee_{k\in J} A_k$ with $\emptyset \neq J \subseteq \{1, \ldots, n\}$. Then c-revisions satisfy (GRK^{strong}) and (GRK^{weak}) in the following sense:*

1. *If $(\lambda_{1,1}^-, \ldots, \lambda_{n_1,1}^-, \lambda_{1,2}^-, \ldots, \lambda_{n_2,2}^-, \ldots, \lambda_{1,n}^-, \ldots, \lambda_{n_n,n}^-, \lambda_S^-)$ defines a c-revision $\kappa * (\mathcal{R} \cup \{S\})$, then each subvector $(\lambda_{1,i}^-, \ldots, \lambda_{n_i,i}^-)$ defines a c-revision $\kappa_{|A_i} * \mathcal{R}_i$ such that $\kappa * (\mathcal{R} \cup \{S\})_{|A_i} = \kappa_{|A_i} * \mathcal{R}_i$.*
2. *If $(\nu_{1,1}^-, \ldots, \nu_{n_1,1}^-, \nu_{1,2}^-, \ldots, \nu_{n_2,2}^-, \ldots, \nu_{1,n}^-, \ldots, \nu_{n_n,n}^-)$ defines a c-revision $\kappa * \mathcal{R}$, then each subvector $(\nu_{1,i}^-, \ldots, \nu_{n_i,i}^-)$ defines a c-revision $\kappa_{|A_i} * \mathcal{R}$ such that $(\kappa * \mathcal{R})_{|A_i} = \kappa_{|A_i} * \mathcal{R}_i$.*
3. *Conversely, if $(\nu_{1,i}^-, \ldots, \nu_{n_i,i}^-)$ defines a c-revision $\kappa_{|A_i} * \mathcal{R}_i$ for each $i = 1, \ldots, n$, then the vector $(\nu_{1,1}^-, \ldots, \nu_{n_1,1}^-, \nu_{1,2}^-, \ldots, \nu_{n_2,2}^-, \ldots, \nu_{1,n}^-, \ldots, \nu_{n_n,n}^-)$ defines a c-revision $\kappa * \mathcal{R}$.*

Proof. Let $\kappa_1^* = \kappa * (\mathcal{R} \cup \{S\})$, $\kappa_2^* = \kappa * \mathcal{R}$ and $\kappa_3^* = \kappa_{|A_i} * \mathcal{R}_i$ for some $i \in \{1, \ldots, n\}$.

We investigate: 1. $\kappa_{1|A_i}^*$ vs. κ_3^*, 2. $\kappa_{2|A_i}^*$ vs. κ_3^* and 3. κ_3^* vs. κ_2^*.

For all $\omega \in \Omega$, $\omega \models A_i$ for exactly one i. If $\omega \models A_i$ then all conditionals from $\mathcal{R}_j$ $(j \neq i)$, are not applicable and hence irrelevant. Therefore, it holds that:

$$\kappa_1^*(\omega) = \kappa * (\mathcal{R} \cup \{S\})(\omega) = \kappa_{1,0} + \kappa(\omega) + \sum_{1\leqslant i\leqslant n} \sum_{\substack{1\leqslant j\leqslant n_i \\ \omega\models A_iC_{j,i}\overline{B}_{j,i}}} \lambda_{j,i}^- + \begin{cases} \lambda_S^- & \omega \not\models S \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and for $i \in \{1, \ldots, n\}$:

$$\lambda_{j,i}^- > \min_{\omega\models A_iC_{j,i}B_{j,i}} \{\kappa(\omega) + \sum_{\substack{l\neq j \\ \omega\models A_iC_{l,i}\overline{B}_{l,i}}} \lambda_{l,i}^-\} - \min_{\omega\models A_iC_{j,i}\overline{B}_{j,i}} \{\kappa(\omega) + \sum_{\substack{l\neq j \\ \omega\models A_iC_{l,i}\overline{B}_{l,i}}} \lambda_{l,i}^-\} \tag{10}$$

λ_S^- occurs only if $i \in J$ and either is cancelled out, or does not appear at all. Furthermore, for $\omega \models A_i$:

$$\kappa_{1|A_i}^*(\omega) = \kappa(\omega) + \sum_{\substack{1\leqslant j\leqslant n_i \\ \omega\models A_iC_{j,i}\overline{B}_{j,i}}} \lambda_{j,i}^- \underbrace{- \min_{\tilde\omega\models A_i} \{\kappa(\tilde\omega) + \sum_{\substack{1\leqslant j\leqslant n_i \\ \tilde\omega\models A_iC_{j,i}\overline{B}_{j,i}}} \lambda_{j,i}^-\}}_{(*)} \tag{11}$$

For each A_i, all models $\omega \models A_i$ either satisfy S or $\overline{S}$, so λ_S^- occurs in both parts of the calculation above or in none of them, and therefore is cancelled out. Also κ_0 is cancelled out.

The definitions of κ_2^* resp. $\kappa_{2|A_i}^*$ and the corresponding impact factors $\nu_{j,i}^-$ follow the same schema as the definitions of κ_1^* resp. $\kappa_{1|A_i}^*$ without the penalty term for λ_S^-. It holds that $\lambda_{j,i}^-$ and $\nu_{j,i}^-$ fulfill the same inequalities. We now turn to κ_3^*:

$$\begin{aligned} \kappa_3^*(\omega) &= \kappa_{|A_i} * \mathcal{R}_i \\ &= \kappa(\omega) + \sum_{\substack{1\leqslant j\leqslant n_i \\ \omega\models A_iC_{j,i}\overline{B}_{j,i}}} \mu_{j,i}^- + \underbrace{\kappa_{3,0} - \kappa(A_i)}_{(**)} \end{aligned} \tag{12}$$

with

$$\mu_{j,i}^- > \min_{\omega\models A_iC_{j,i}B_{j,i}} \{\kappa(\omega) + \sum_{\substack{l\neq j \\ \omega\models A_iC_{l,i}\overline{B}_{l,i}}} \mu_{l,i}^-\} - \min_{\omega\models A_iC_{j,i}\overline{B}_{j,i}} \{\kappa(\omega) + \sum_{\substack{l\neq j \\ \omega\models A_iC_{l,i}\overline{B}_{l,i}}} \mu_{l,i}^-\} \tag{13}$$

We have (10)=(13), and Lemma 1 yields

$$\kappa_{3,0} = -\min_{\tilde\omega\models A_i} \{\kappa(\tilde\omega) + \sum_{1\leqslant j\leqslant n_i, \omega\models A_iC_{j_i}\overline{B}_{j_i}} \mu_{j,i}^-\} = (*) + \kappa(A_i),$$

so $(*) = (**)$. Therefore the first two statements of the theorem are proved. The third statement follows immediately because the impact factors $\nu_{j,i}^-$ for each subvector fulfill the success condition of κ_2^*. Note that these benefits are due to the specific schema of irrelevance and cancellations among the respective conditional impacts. □

S represents new evidence affecting the epistemic state of an agent. For $\mathcal{R} = \emptyset$, (GRK^{strong}) induces that we strengthen the beliefs of the cases A_i in S, without changing the respective conditional beliefs. This corresponds to Spohn's *law of generalized conditionalization* (Spohn 2014) which is the analogon of Jeffrey's rule in the framework of ranking functions.

We give an example of (GRK^{weak}) for c-revisions..

Example 5. *Let $\Sigma = \{a, b, c, d\}$ and $\mathcal{R}_1 = \{(c|ab), (d|abc)\}$, $\mathcal{R}_2 = \{(d|a\overline{b})\}$, $\mathcal{R}_3 = \{(d|\overline{a}), (cd|\overline{a})\}$ such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ and $\mathcal{P}_{\mathcal{R}} = \{ab, a\overline{b}, \overline{a}\}$ as the*

$\omega \in \Omega$	κ	$\kappa * \mathcal{R}$	$(\kappa * \mathcal{R})^{min}$	$(\kappa * \mathcal{R})^{min}_{\vert A_i}$
$abcd$	5	-1 + 5	4	0
$abc\overline{d}$	2	-1 + 2 + α_2^-	5	1
$ab\overline{c}d$	3	-1 + 3 + α_1^-	5	1
$ab\overline{c}\overline{d}$	4	-1 + 4 + α_1^-	6	2
$a\overline{b}cd$	3	-1 + 3	2	2
$a\overline{b}c\overline{d}$	0	-1 + 0 + β_1^-	1	1
$a\overline{b}\overline{c}d$	1	-1 + 1	0	0
$a\overline{b}\overline{c}\overline{d}$	2	-1+ 2 + β_1^-	3	3
$\overline{a}bcd$	4	-1 + 4	3	0
$\overline{a}bc\overline{d}$	1	-1 + 1 + γ_1^- + γ_2^-	4	1
$\overline{a}b\overline{c}d$	2	-1 + 2 + γ_2^-	5	2
$\overline{a}b\overline{c}\overline{d}$	3	-1 + 3 +γ_1^- + γ_2^-	6	3
$\overline{a}\overline{b}cd$	6	-1 + 6	5	2
$\overline{a}\overline{b}c\overline{d}$	3	-1 + 3 + γ_1^- + γ_2^-	6	3
$\overline{a}\overline{b}\overline{c}d$	4	-1 + 4 + γ_2^-	7	4
$\overline{a}\overline{b}\overline{c}\overline{d}$	5	-1 + 5 + γ_1^- + γ_2^-	8	5
κ_0		-1		

Table 1: The table displays the OCF κ and the c-revised (and conditionalized) OCF $\kappa * \mathcal{R}$ resp. $(\kappa * \mathcal{R})_{\vert A_i}$ both as a schema and with pareto-minimal impact factors.

$\omega \in \Omega$	$\kappa_{\vert A_i}$	$(\kappa^{*1}_{\vert A_1})^{min}$	$(\kappa^{*2}_{\vert A_2})^{min}$	$(\kappa^{*3}_{\vert A_3})^{min}$
$abcd$	3	0		
$abc\overline{d}$	0	1		
$ab\overline{c}d$	1	1		
$ab\overline{c}\overline{d}$	2	2		
$a\overline{b}cd$	3		2	
$a\overline{b}c\overline{d}$	0		1	
$a\overline{b}\overline{c}d$	1		0	
$a\overline{b}\overline{c}\overline{d}$	2		3	
$\overline{a}bcd$	3			0
$\overline{a}bc\overline{d}$	0			1
$\overline{a}b\overline{c}d$	1			2
$\overline{a}b\overline{c}\overline{d}$	2			3
$\overline{a}\overline{b}cd$	5			2
$\overline{a}\overline{b}c\overline{d}$	2			3
$\overline{a}\overline{b}\overline{c}d$	3			4
$\overline{a}\overline{b}\overline{c}\overline{d}$	4			5
κ_0		-3	-1	-3

Table 2: Conditionalized OCF $\kappa_{\vert A_i}$ and the firstly conditionalized and then c-revised OCFs $\kappa_{\vert A_i} * \mathcal{R}_i$ with $i = 1, 2, 3$, notated as $\kappa^{*i}_{\vert A_i}$. We use the Pareto-minimal impact factors.

finest premise splitting. The OCF κ can be found in table 1, along with a schematic c-revised $\kappa * \mathcal{R}$ and the c-revised OCF with (Pareto-)minimal parameters $\alpha_1^- = 3, \alpha_2^- = 4, \beta_1^- = 2, \gamma_1^- = 0, \gamma_2^- = 4$ belonging to the conditionals in $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, respectively. In table 2 the c-revisions with conditionalized $\kappa_{\vert A_i}$ are displayed, we chose again the (Pareto-)minimal parameters $(\kappa * \mathcal{R}_{\vert A_i})_{min}$. If we compare the conditionalized version with the c-revisions from table 2, it is clear that $\kappa_{\vert A_i} * \mathcal{R}_i = \kappa * \mathcal{R}_{\vert A_i}$.

6 Conclusion

Generalized Ranking Kinematics (GRK) aims to capture the intuition that information concerning exclusive cases should be revised independently. Jeffrey (Jeffrey 1965) implemented this idea in the probabilistic framework for sets of propositions. Shore and Johnson (Shore and Johnson 1980) extended this notion for sets of conditionals and showed that the ability to differentiate between different contexts is crucial for inductive inference in the probabilistic framework. In this paper we have shown that Shore and Johnsons' ideas are applicable to ranking functions. The key to our approach is to split the premises of the set of conditionals $\mathcal{R}$. We provided an algorithm to compute the premise splitting for an arbitrary finite set of conditionals and proved that we obtain the finest premise splitting, which maximises the benefits of GRK. The main motivation for studying the concept of Probability Kinematics resp. Subset Independence was to set up local contexts for revision with sets of conditionals for the framework of ranking functions. GRK allows us to set up local contexts also for qualitative revision.

In (Kern-Isberner 2001) c-revisions were devised as a qualitative counterpart to probabilistic revision via the principle of minimum cross-entropy (MinCEnt) and thus inherit many qualities of that revision operator. As we have mentioned above GRK is inspired by Subset Independence which is one of the characterizing axioms of MinCEnt, according to Shore and Johnsons (Shore and Johnson 1980). As shown in this paper, c-revisions fulfill GRK and hence allow us to implement this powerful principle for qualitative iterated revision.

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