The Thirty-Second International Florida
Artificial Intelligence Research Society Conference (FLAIRS-32)

On Rational Monotony and Weak Rational Monotony for Inference Relations Induced by Sets of Minimal C-Representations

Christoph Beierle, Steven Kutsch, Henning Breuers
Department of Computer Science, FernUniversität in Hagen, Hagen, Germany

Abstract

Reasoning in the context of a conditional knowledge base containing rules of the form 'If A then usually B' can be defined in terms of preference relations on possible worlds. These preference relations can be modeled by ranking functions that assign a degree of disbelief to each possible world. In general, there are multiple ranking functions that accept a given knowledge base. Several nonmonotonic inference relations have been proposed using c-representations, a subset of all ranking functions. These inference relations take subsets of all c-representations based on various notions of minimality into account, and they operate in different inference modes, i.e., skeptical, weakly skeptical, or credulous. For nonmonotonic inference relations, weaker versions of monotonicity like rational monotony (RM) and weak rational monotony (WRM) have been developed. In this paper, we investigate which of the inference relations induced by sets of minimal c-representations satisfy rational monotony or weak rational monotony.

1 Introduction

Using qualitative conditionals of the form "If A, then usually B", formally denoted by $(B|A)$, is a standard approach for defining nonmonotonic inference relations. In order to give a semantics to such conditionals, the semantic methods of classical logic are not sufficient, and richer structures are needed. Popular semantics for conditional logic are, e.g., Lewis' system of spheres (Lewis 1973), conditional objects evaluated using boolean intervals (Dubois and Prade 1994), possibility distributions (Benferhat, Dubois, and Prade 1999), or ranking functions (Spohn 2012). Particular ranking functions using the principle of conditional preservation as their core construction mechanism are c-representations (Kern-Isberner 2001; 2004).

Each single c-representation exhibits desirable inference properties (see e.g. (Thorn et al. 2015)), and it was shown that skeptical c-inference over all c-representations satisfies and exceeds system P (Beierle et al. 2018). Both as approximations and as liberalizations of skeptical c-inference, various inference relations taking different minimal classes of c-representations into account have been proposed; moreover, these inference relations operate in three different inference modes, namely skeptical, weakly skeptical and credulous (Beierle et al. 2016; 2018). For these inference relations, the well-known postulates of Rational Monotony (RM), on the agenda since the early days of nonmonotonic reasoning, and its variant Weak Rational Monotony (WRM) have not yet been investigated for c-inference relations; only for skeptical c-inference over all c-representations it is known that (RM) is not satisfied [1]. For all other c-inference relations, in particular over minimal c-representations, their properties with respect to (RM) and (WRM) have been open problems. This paper addresses these open questions and provides the following main contributions:

- Skeptical inference over any notion of minimal c-representations does not satisfy (RM), but satisfies (WRM).
- Weakly skeptical inference over all c-representations and over any notion of minimal c-representations does not satisfy (RM), but satisfies (WRM).
- Credulous inference over all c-representations and over any notion of minimal c-representations satisfies both (RM) and (WRM).

2 Background

Conditional Logic and OCFs Let $\mathcal{L}$ be a propositional language over a finite set Σ of atoms $a, b, c, \ldots$. The formulas of $\mathcal{L}$ will be denoted by letters $A, B, C, \ldots$. We write AB for $A \wedge B$ and $\overline{A}$ for $\neg A$. We identify the set of all complete conjunctions over Σ with the set Ω of possible worlds over $\mathcal{L}$. For $\omega \in \Omega$, $\omega \models A$ means that $A \in \mathcal{L}$ holds in ω.

By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. $(B|A)$ formalizes "*if A then usually B*" and establishes a plausible connection between the *antecedent* A and the *consequence* B. A conditional $(B|A)$ partitions the set of worlds Ω in three parts: those worlds satisfying AB, thus *verifying* the conditional, those worlds satisfying $A\overline{B}$, thus *falsifying* the conditional, and those worlds not fulfilling the premise A and so which the conditional may not be applied to at all (de Finetti 1937).

[1] The first counterexample to (RM) of skeptical c-inference was developed by Hans Rott, when discussing c-representations and c-inference with him.

To give appropriate semantics to conditionals, they are usually considered within richer structures such as *epistemic states*. In this paper, we consider Spohn's *ordinal conditional functions, OCFs* (Spohn 2012). An OCF is a function $\kappa : \Omega \to \mathbb{N}$ expressing degrees of plausibility of possible worlds where a higher degree denotes "less plausible" or "more surprising". At least one world must be regarded as being normal; therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. Each such κ can be taken as the representation of a full epistemic state of an agent, and it uniquely extends to a function (also denoted by κ) mapping sentences to $\mathbb{N} \cup \{\infty\}$ by:

$$\kappa(A) = \begin{cases} \min\{\kappa(\omega) \mid \omega \models A\} & \text{if } A \text{ is satisfiable} \\ \infty & \text{otherwise} \end{cases} \quad (1)$$

For all ranking functions κ and all formulas A it holds that

$$\kappa(A) = 0 \quad \text{or} \quad \kappa(\overline{A}) = 0. \quad (2)$$

An OCF κ *accepts* a conditional $(B|A)$ (denoted by $\kappa \models (B|A)$) if the verification of the conditional is less surprising than its falsification, i.e., if $\kappa(AB) < \kappa(A\overline{B})$. A non-empty finite set $\mathcal{R} \subseteq (\mathcal{L}|\mathcal{L})$ of conditionals is called a *knowledge base* if it does not contain any self-fulfilling ($A \models B$) or contradictory ($A \models \overline{B}$) conditional. An OCF κ accepts $\mathcal{R}$ if κ accepts all conditionals in $\mathcal{R}$, and $\mathcal{R}$ is *consistent* if an OCF accepting $\mathcal{R}$ exists (Goldszmidt and Pearl 1996).

C-Representations Different ways of determining a ranking function for a knowledge base $\mathcal{R}$ have been proposed, e.g., the well-known *system Z* (Pearl 1990). Here, we will use *c-representations* that assign an individual impact η_i to each conditional $(B_i|A_i)$ and generate the world ranks as a sum of impacts of falsified conditionals:

Definition 1 (c-representation (Kern-Isberner 2001; 2004))**.** *A* c-representation *of a knowledge base $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ is an OCF κ constructed from non-negative integer impacts $\eta_i \in \mathbb{N}_0$ assigned to each conditional $(B_i|A_i)$ such that κ accepts $\mathcal{R}$ and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \quad (3)$$

C-representations can conveniently be specified using a constraint satisfaction problem (for detailed explanations, see (Kern-Isberner 2001; 2004; Beierle et al. 2018)):

Definition 2 ($CR(\mathcal{R})$ (Beierle et al. 2018))**.** *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:*

$$\eta_i \geqslant 0 \quad (4)$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \quad (5)$$

A solution of $CR(\mathcal{R})$ is a vector $(\eta_1, \ldots, \eta_n)$ of n natural numbers. For a constraint satisfaction problem CSP, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR(\mathcal{R}))$ we denote the set of all solutions of $CR(\mathcal{R})$. For $\vec{\eta} \in Sol(CR(\mathcal{R}))$ and κ as in Equation (3), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$.

Proposition 3 (Soundness and Completeness of $CR(\mathcal{R})$ (Beierle et al. 2018))**.** *Let $\mathcal{R}$ be a knowledge base. For every $\vec{\eta} \in Sol(CR(\mathcal{R}))$, $\kappa_{\vec{\eta}}$ is a c-representation with $\kappa_{\vec{\eta}} \models \mathcal{R}$, and for every c-representation κ with $\kappa \models \mathcal{R}$, there is a $\vec{\eta} \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa_{\vec{\eta}}$.*

3 Inference With OCFs and Sets of OCFs

In (Goldszmidt and Pearl 1996), the nonmonotonic inference relation between a premise A and a conclusion B induced by an OCF κ, written $A \mid\!\sim_\kappa B$, is fully identified with the acceptance of the conditional $(B|A)$ by κ. However, contrary to the claim stated in (Goldszmidt and Pearl 1996, p. 66), as well as in many other publications, p-entailment based on $\mid\!\sim_\kappa$ defined in this way does not satisfy system P (Lehmann and Magidor 1992). The inference rule (REF) (i.e. $A \mid\!\sim A$) in system P does not hold for this definition of $\mid\!\sim_\kappa$, and hence also not for p-entailment based on it, because $\bot \not\mid\!\sim_\kappa \bot$.

Hence, we use an alternative definition of $\mid\!\sim_\kappa$ given by:

$$A \mid\!\sim_\kappa B \quad \text{iff} \quad A \equiv \bot \quad \text{or} \quad \kappa(AB) < \kappa(A\overline{B}) \quad (6)$$

Whereas Equation (6) defines an inference relation $\mid\!\sim_\kappa$ based on a single OCF κ, skeptical, weakly skeptical, and credulous c-inference take all c-representations of a given knowledge base $\mathcal{R}$ into account.

Definition 4 (skeptical, weakly skeptical, and credulous c-inference (Beierle et al. 2016; 2018))**.** *Let $\mathcal{R}$ be a knowledge base and let A, B be formulas.*

1. *B is* a skeptical c-inference from A in the context of *$\mathcal{R}$, denoted by $A \mid\!\sim^{sk}_{\mathcal{R}} B$, if $A \mid\!\sim_\kappa B$ holds for all c-representations κ for $\mathcal{R}$.*
2. *B is a* weakly skeptical c-inference from A in the context of *$\mathcal{R}$, denoted by $A \mid\!\sim^{ws}_{\mathcal{R}} B$, if $A \equiv \bot$, or there is a c-representation κ for $\mathcal{R}$ such that $A \mid\!\sim_\kappa B$ holds and there is no c-representation κ' for $\mathcal{R}$ such that $A \mid\!\sim_{\kappa'} \overline{B}$.*
3. *B is a* credulous c-inference from A in the context of $\mathcal{R}$, *denoted by $A \mid\!\sim^{cr}_{\mathcal{R}} B$, if there is a c-representation κ for $\mathcal{R}$ such that $A \mid\!\sim_\kappa B$ holds.*

Weakly skeptical inference lies indeed strictly between skeptical and credulous inference.

Proposition 5 ((Beierle et al. 2016))**.** *For every consistent knowledge base $\mathcal{R}$ we have*

$$\mid\!\sim^{sk}_{\mathcal{R}} \;\subsetneq\; \mid\!\sim^{ws}_{\mathcal{R}} \;\subsetneq\; \mid\!\sim^{cr}_{\mathcal{R}} \quad (7)$$

and there is an $\mathcal{R}$ such that the inclusions in (7) *are strict.*

From the point of view of minimal specificity, c-representations yielding minimal degrees of implausibility are most interesting. Different orderings on $Sol(CR(\mathcal{R}))$ leading to different minimality notions can be used.

Definition 6 (sum-, cw-, ind-minimal (Beierle et al. 2016))**.** *Let $\mathcal{R}$ be a knowledge base and $\vec{\eta}$, $\vec{\eta}' \in Sol(CR(\mathcal{R}))$ with $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ and $\vec{\eta}' = (\eta'_1, \ldots, \eta'_n)$.*

$$\vec{\eta} \preccurlyeq_+ \vec{\eta}' \quad \textit{iff} \quad \sum_{1 \leqslant i \leqslant n} \eta_i \leqslant \sum_{1 \leqslant i \leqslant n} \eta'_i \quad (8)$$

ω	r_1	r_2	r_3	r_4	r_5	r_6	imp. on ω	$\kappa_{\vec{\eta}_1}$	$\kappa_{\vec{\eta}_2}$	$\kappa_{\vec{\eta}_3}$	$\kappa_{\vec{\eta}_4}$
$bcst$	v	v	f	$-$	f	f	$\eta_3{+}\eta_5{+}\eta_6$	5	5	11	16
$bcs\overline{t}$	v	f	f	f	$-$	v	$\eta_2{+}\eta_3{+}\eta_4$	3	3	6	6
$bc\overline{s}t$	v	v	f	$-$	v	$-$	η_3	1	1	1	4
$bc\overline{s}\overline{t}$	v	f	f	f	$-$	$-$	$\eta_2{+}\eta_3{+}\eta_4$	3	3	6	6
$b\overline{c}st$	f	v	$-$	$-$	f	$-$	$\eta_1 + \eta_5$	1	2	7	7
$b\overline{c}s\overline{t}$	f	f	$-$	v	$-$	$-$	$\eta_1 + \eta_2$	2	2	5	3
$b\overline{c}\overline{s}t$	f	v	$-$	$-$	v	$-$	η_1	1	1	2	1
$b\overline{c}\overline{s}\overline{t}$	f	f	$-$	v	$-$	$-$	$\eta_1 + \eta_2$	2	2	5	3
$\overline{b}cst$	v	v	v	$-$	f	f	$\eta_5 + \eta_6$	4	4	10	12
$\overline{b}cs\overline{t}$	v	f	v	$-$	$-$	f	$\eta_2 + \eta_6$	5	4	8	8
$\overline{b}c\overline{s}t$	v	v	v	$-$	v	$-$	0	0	0	0	0
$\overline{b}c\overline{s}\overline{t}$	v	f	v	$-$	$-$	$-$	η_2	1	1	3	2
$\overline{b}\overline{c}st$	f	v	$-$	$-$	f	$-$	$\eta_1 + \eta_5$	1	2	7	7
$\overline{b}\overline{c}s\overline{t}$	f	f	$-$	$-$	$-$	$-$	$\eta_1 + \eta_2$	2	2	5	3
$\overline{b}\overline{c}\overline{s}t$	f	v	$-$	$-$	v	$-$	η_1	1	1	2	1
$\overline{b}\overline{c}\overline{s}\overline{t}$	f	f	$-$	$-$	$-$	$-$	$\eta_1 + \eta_2$	2	2	5	3

imp.	η_1	η_2	η_3	η_4	η_5	η_6	
$\vec{\eta}_1$	1	1	1	1	0	4	*cw-, sum-, ind-minimal*
$\vec{\eta}_2$	1	1	1	1	1	3	*cw-, sum-, ind-minimal*
$\vec{\eta}_3$	2	3	1	2	5	5	
$\vec{\eta}_4$	1	2	4	0	6	6	

Table 1: Verification and falsification behaviour for $\mathcal{R} = \{r_1{:}(c|\top), r_2{:}(t|\top), r_3{:}(\overline{b}|c), r_4{:}(\overline{c}|b\overline{t}), r_5{:}(\overline{s}|t), r_6{:}(b\overline{t}|cs)\}$ from Example 7 and four accepting c-representations.

$\vec{\eta}$ *is* sum-minimal *iff* $\vec{\eta} \preccurlyeq_+ \vec{\eta}'$ *for all* $\vec{\eta}' \in Sol(CR(\mathcal{R}))$. *We write* $\vec{\eta} \prec_+ \vec{\eta}'$ *iff* $\vec{\eta} \preccurlyeq_+ \vec{\eta}'$ *and* $\vec{\eta}' \not\preccurlyeq_+ \vec{\eta}$.

$$\vec{\eta} \preccurlyeq_{cw} \vec{\eta}' \quad \textit{iff} \quad \eta_i \leqslant \eta_i' \textit{ for all } i \in \{1, \ldots, n\} \tag{9}$$

$\vec{\eta}$ *is* cw-minimal *(or* Pareto-minimal*) iff there is no vector* $\vec{\eta}' \in Sol(CR(\mathcal{R}))$ *such that* $\vec{\eta}' \preccurlyeq_{cw} \vec{\eta}$ *and* $\vec{\eta} \not\preccurlyeq_{cw} \vec{\eta}'$.

$$\vec{\eta} \preccurlyeq_O \vec{\eta}' \quad \textit{iff} \quad \kappa_{\vec{\eta}}(\omega) \leqslant \kappa_{\vec{\eta}'}(\omega) \textit{ for all } \omega \in \Omega \tag{10}$$

$\vec{\eta}$ *is* ind-minimal *iff there is no vector* $\vec{\eta}' \in Sol(CR(\mathcal{R}))$ *such that* $\vec{\eta}' \preccurlyeq_O \vec{\eta}$ *and* $\vec{\eta} \not\preccurlyeq_O \vec{\eta}'$.

Thus, while sum-minimal and cw-minimal are defined by just taking the components of the solution vectors $\vec{\eta}$ into account, ind-minimality refers to the ranking function induced by a solution vector. The following example for these concepts will be used in subsequent propositions.

Example 7. *For* $\mathcal{R} = \{r_1 : (c|\top), r_2 : (t|\top), r_3 : (\overline{b}|c), r_4 : (\overline{c}|b\overline{t}), r_5 : (\overline{s}|t), r_6 : (b\overline{t}|cs)\}$*, Table 1 shows the verification and falsification for every possible world* ω*. Applying some optimization rules to* $CR(\mathcal{R})$ *(Beierle, Kutsch, and Sauerwald 2018), yields the simplified CSP*

$$\begin{array}{ll} \eta_1 > 0 & \eta_4 > \eta_1 - \eta_3 \\ \eta_2 > 0 & \eta_5 > 0 - \min\{\eta_1, \eta_6\} \\ \eta_3 > 0 & \eta_6 > \eta_2 + \eta_3 + \eta_4 - \min\{\eta_2, \eta_5\}\,. \end{array} \tag{11}$$

Table 1 also shows four solutions $\vec{\eta}_1$*,* $\vec{\eta}_2$*,* $\vec{\eta}_3$*, and* $\vec{\eta}_4$ *of* $CR(\mathcal{R})$ *with their induced c-representations* $\kappa_{\vec{\eta}_i}$*. Note that* $\vec{\eta}_1$ *and* $\vec{\eta}_2$ *are cw-, sum-, and ind-minimal while* $\vec{\eta}_3$ *and* $\vec{\eta}_4$ *are not minimal with respect to any of the three notions of minimality.*

Viewing the models that are minimal with respect to one of the ordering relations given in Definition 6 as preferred models, skeptical, credulous, and weakly skeptical inference versions can be obtained from the definitions of $\mathrel{|\!\!\sim}^{sk}_{\mathcal{R}}$, $\mathrel{|\!\!\sim}^{cr}_{\mathcal{R}}$, and $\mathrel{|\!\!\sim}^{ws}_{\mathcal{R}}$ by replacing the set $Sol(CR(\mathcal{R}))$ by the respective set of minimal solutions. We first define the three different inference modes over arbitrary sets of ranking models.

Definition 8. *Let* M *be a set of OCFs accepting a knowledge base* $\mathcal{R}$*, and let* A *and* B *be formulas.*

1. $A \mathrel{|\!\!\sim}^{sk,M}_{\mathcal{R}} B$ *if* $A \mathrel{|\!\!\sim}_{\kappa} B$ *for all* $\kappa \in M$
2. $A \mathrel{|\!\!\sim}^{ws,M}_{\mathcal{R}} B$ *if* $A \equiv \bot$*, or there is a* $\kappa \in M$ *such that* $A \mathrel{|\!\!\sim}_{\kappa} B$ *and there is no* $\kappa' \in M$ *such that* $A \mathrel{|\!\!\sim}_{\kappa'} \overline{B}$
3. $A \mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} B$ *if there is a* $\kappa \in M$ *such that* $A \mathrel{|\!\!\sim}_{\kappa} B$

Definition 9 (min-inference)**.** *Let* $\mathcal{R}$ *be a knowledge base, let* A, B *be formulas, and let* $\bullet \in \{+, cw, O\}$*. Furthermore, let* $M_\bullet = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR(\mathcal{R}))$ *and* $\vec{\eta}$ *is* $\bullet$*-minimal*$\}$.

1. B *is* a skeptical $\bullet$-min-inference from A in the context of $\mathcal{R}$*, denoted by* $A \mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} B$*, if* $A \mathrel{|\!\!\sim}^{sk,M_\bullet}_{\mathcal{R}} B$.
2. B *is a* weakly skeptical $\bullet$-min-inference from A in the context of $\mathcal{R}$*, denoted by* $A \mathrel{|\!\!\sim}^{ws,\bullet}_{\mathcal{R}} B$*, if* $A \mathrel{|\!\!\sim}^{ws,M_\bullet}_{\mathcal{R}} B$.
3. B *is a* credulous $\bullet$-min-inference from A in the context of $\mathcal{R}$*, denoted by* $A \mathrel{|\!\!\sim}^{cr,\bullet}_{\mathcal{R}} B$*, if* $A \mathrel{|\!\!\sim}^{cr,M_\bullet}_{\mathcal{R}} B$.

The inclusions given in Prop. 5 carry over to all three kinds of min-inference.

Proposition 10 ((Beierle et al. 2016))**.** *For every consistent knowledge base* $\mathcal{R}$ *and* $\bullet \in \{+, cw, O\}$ *we have*

$$\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} \;\subseteq\; \mathrel{|\!\!\sim}^{ws,\bullet}_{\mathcal{R}} \;\subseteq\; \mathrel{|\!\!\sim}^{cr,\bullet}_{\mathcal{R}} \tag{12}$$

and there is an $\mathcal{R}$ *such that the inclusions in* (12) *are strict.*

Propositions 5 and 10 state only some of the interrelationships among the different inference relations based on c-representations; for a detailed study of further interrelationships we refer to (Beierle et al. 2016).

4 Rational Monotony

In this section, we investigate the various c-inference relations with respect to the concept of rational monotony (Lehmann and Magidor 1992).

Definition 11 (Rational Monotony)**.** *The nonmonotonic inference relation* $\mathrel{|\!\!\sim}$ *satisfies* Rational Monotony (RM) *if for all propositional formulas* A*,* B *and* C*, it holds that*

$$\frac{A \mathrel{|\!\!\sim} B \qquad A \mathrel{|\!\!\not\sim} \overline{C}}{AC \mathrel{|\!\!\sim} B} \tag{RM}$$

As already mentionend in Sec.1, it is known that skeptical inference over all c-representations does not satisfy (RM). However, it has been unknown whether skeptical and/or weakly skeptical inference over any of the introduced sets of minimal c-representations does satisfy (RM). It also has been unknown whether credulous inference over any minimal set of c-representations satisfies (RM).

Proposition 12. *Credulous inference over any set M of c-representations satisfies* (RM).

Proof. Let $\mathcal{R}$ be a knowledge base and M be any set of c-representations accepting $\mathcal{R}$. It is well known that preferential inference over any ranking model satisfies (RM) (Lehmann and Magidor 1992). We show that if both $A \mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} B$ and $A \not\mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} \overline{C}$ hold, it also holds that

$$AC \mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} B. \tag{13}$$

Since $A \not\mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} \overline{C}$ holds, for every c-representation κ in M it holds that

$$A \not\mathrel{|\!\!\sim}_{\kappa} \overline{C}. \tag{14}$$

Since $A \mathrel{|\!\!\sim}^{cr,M}_{\mathcal{R}} B$ holds, there is a c-representation κ' in M such that

$$A \mathrel{|\!\!\sim}_{\kappa'} B \tag{15}$$

holds. Because every $\mathrel{|\!\!\sim}_{\kappa}$ satisfies (RM), from instantiating (14) with κ' and using (15) it follows that

$$AC \mathrel{|\!\!\sim}_{\kappa'} B$$

and therefore also (13) holds. □

Note that even though credulous inference does satisfy (RM), it is generally too bold because it easily leads to inconsistencies, and it is still unknown whether it satisfies other desirable properties of nonmonotonic inference relations, such as (OR), (CUT) or (CM) (Lehmann and Magidor 1992).

The following is an example of a skeptical c-inference.

Example 13. *For the knowledge base $\mathcal{R}$ from Example 7 it holds that $b \mathrel{|\!\!\sim}^{sk}_{\mathcal{R}} t$.*

We have to show that for all c-representations κ with $\kappa \models \mathcal{R}$ it holds that $\kappa(bt) < \kappa(b\overline{t})$. From the falsification of conditionals listed in Table 1, we get the ranks of the four worlds satisfying bt for all c-representations κ as

$$\kappa(bcst) = \eta_3 + \eta_5 + \eta_6 \qquad \kappa(bc\overline{s}t) = \eta_3$$
$$\kappa(b\overline{c}st) = \eta_1 + \eta_5 \qquad \kappa(b\overline{c}\overline{s}t) = \eta_1.$$

Since all impacts η_i are non-negative, the rank of bt for all c-representations κ is given as $\kappa(bt) = min\,\{\eta_1, \eta_3\}$. For the four worlds that satisfy $b\overline{t}$ we get for all c-representations κ

$$\kappa(bcs\overline{t}) = \eta_2 + \eta_3 + \eta_4 \qquad \kappa(bc\overline{s}\overline{t}) = \eta_2 + \eta_3 + \eta_4$$
$$\kappa(b\overline{c}s\overline{t}) = \eta_1 + \eta_2 \qquad \kappa(b\overline{c}\overline{s}\overline{t}) = \eta_1 + \eta_2.$$

The rank of $b\overline{t}$ is therefore in all c-representations κ given as $\kappa(b\overline{t}) = min\,\{\eta_1 + \eta_2, \eta_2 + \eta_3 + \eta_4\}$. For showing that in all c-representations κ it holds that $\kappa(bt) < \kappa(b\overline{t})$, we distinguish two cases:

Case 1 *$\eta_1 \leqslant \eta_3$: In this case we have $\kappa(bt) = min\,\{\eta_1, \eta_3\} = \eta_1$. Since the constraint for η_2 is given as $\eta_2 > 0$ (Example 7), we know that $\eta_1 < \eta_1 + \eta_2$. Since $\eta_1 \leqslant \eta_3$ it must then also hold that $\kappa(b\overline{t}) = min\,\{\eta_1 + \eta_2, \eta_2 + \eta_3 + \eta_4\} = \eta_1 + \eta_2$ and finally $\kappa(bt) < \kappa(b\overline{t})$.*

Case 2 *$\eta_1 > \eta_3$: In this case we get for bt*

$$\kappa(bt) = min\,\{\eta_1, \eta_3\} = \eta_3. \tag{16}$$

In Example 7 the constraint for η_4 is given as $\eta_4 > \eta_1 - \eta_3$. Therefore, it also holds that $\eta_3 + \eta_4 > \eta_1$ and also $\eta_2 + \eta_3 + \eta_4 > \eta_1 + \eta_2$ and finally

$$\kappa(b\overline{t}) = min\,\{\eta_1 + \eta_2, \eta_2 + \eta_3 + \eta_4\} = \eta_1 + \eta_2. \tag{17}$$

From (16) *and* (17) *and since $\eta_1 > \eta_3$ we get $\kappa(bt) = \eta_3 < \eta_1 + \eta_2 = \kappa(b\overline{t})$. This shows that $b \mathrel{|\!\!\sim}^{sk}_{\mathcal{R}} t$ holds.*

Proposition 14. *Skeptical inference $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$ over any of the three sets of minimal c-representations does not satisfy* (RM).

Proof. To show that skeptical inference over a set of minimal c-representations does not satisfy (RM), we have to choose a knowledge base $\mathcal{R}$ and identify three formulas A, B and C such that $A \mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} B$ and $A \not\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} \overline{C}$ but also $AC \not\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} B$ hold where $\bullet \in \{cw, +, O\}$. We choose $\mathcal{R}$ from Example 7 and

$$A = b \qquad B = t \qquad C = \overline{c} \wedge s.$$

This means we have to show that

$$b \mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} t \tag{18}$$
$$b \not\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} c \vee \overline{s} \tag{19}$$
$$b \wedge \overline{c} \wedge s \not\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} t. \tag{20}$$

For (18)**:** Example 13 states, that $b \mathrel{|\!\!\sim}^{sk}_{\mathcal{R}} t$ holds. Since $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$ is skeptical inference over a subset of c-representations, it also holds that $b \mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} t$ for $\bullet \in \{cw, O, +\}$.

For (19)**:** To show that (19) holds we need to show that there is a c-representation κ that is minimal with respect to the minimality criterion $\bullet$ such that $\kappa(b \wedge (c \vee \overline{s})) \geqslant \kappa(b\overline{c}s)$ holds. The c-representation $\kappa_{\vec{\eta}_1}$ from Table 1 is cw-, sum- and ind-minimal. It holds that

$$\kappa_{\vec{\eta}_1}(b \wedge (c \vee \overline{s})) = min\,\{1, 1\} \geqslant min\,\{2, 1\} = \kappa_{\vec{\eta}_1}(b\overline{c}s)$$

Therefore, (19) holds. We now know that the two prerequisites of (RM) as given in (18) and (19) hold for $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$ with $\bullet \in \{cw, +, O\}$.

For (20)**:** We have to show that there is a c-representation κ that is minimal with respect to $\bullet$ such that $\kappa(b\overline{c}st) \geqslant \kappa(b\overline{c}s\overline{t})$. The c-representation $\kappa_{\vec{\eta}_2}$ from Table 1 is cw-, sum- and ind-minimal. It holds that

$$\kappa_{\vec{\eta}_2}(b\overline{c}st) = 2 \geqslant 2 = \kappa_{\vec{\eta}_2}(b\overline{c}s\overline{t})$$

With this we have proven (18), (19) and (20). Therefore rational monotony does not hold for skeptical inference over any of the three sets of minimal c-representations. □

Note that due to Example 13 the relation in (18) holds for skeptical c-inference and that the relations in (19) and (20) also hold with respect to skeptical c-inference since the sets of minimal models are subsets of all c-representations. Therefore, the counterexample we used in Proposition 14 for proving that (RM) does not hold for skeptical c-inference over a set of minimal c-representations provides another counterexample for (RM) for skeptical c-inference over the set of all c-representations.

Regarding (RM), we now turn to weakly skeptical inference.

Proposition 15. *Weakly skeptical inference over all c-representation does not satisfy* (RM).

Proof. Again we use the knowledge base $\mathcal{R}$ from Example 7. In order to show that (RM) does not hold for the weakly skeptical inference relation $\mid\!\sim^{ws}_{\mathcal{R}}$, we need to identify three formulas A, B and C such that $A \mid\!\sim^{ws}_{\mathcal{R}} B$ and $A \not\mid\!\sim^{ws}_{\mathcal{R}} \overline{C}$ but also $AC \mid\!\sim^{ws}_{\mathcal{R}} B$ hold. We choose

$$A = b \qquad B = t \qquad C = c \vee s.$$

This means we need to show that

$$b \;\mid\!\sim^{ws}_{\mathcal{R}}\; t \tag{21}$$
$$b \;\not\mid\!\sim^{ws}_{\mathcal{R}}\; \overline{c \vee s} \tag{22}$$
$$b \wedge (c \vee s) \;\not\mid\!\sim^{ws}_{\mathcal{R}}\; t \tag{23}$$

hold.

For (21)**:** By Example 13 we know that $b \mid\!\sim^{sk}_{\mathcal{R}} t$. With Proposition 5 we get that (21) holds.

For (22)**:** The impact vector $\vec{\eta}_3 = (2, 3, 1, 2, 5, 5)$ is a solution for the CSP (11) given in Example 7, so the c-representation $\kappa_{\vec{\eta}_3}$ (cf. Table 1) is a c-representation accepting $\mathcal{R}$. Since it holds that

$$\kappa_{\vec{\eta}_3}(b\,\overline{c}\,\overline{s}) = 2 > 1 = \kappa_{\vec{\eta}_3}(b \wedge (c \vee s))$$

we get $b \mid\!\sim_{\kappa_{\vec{\eta}_3}} c \vee s$, which implies (22).

For (23)**:** Similarly, the impact vector $\vec{\eta}_4 = (1, 2, 4, 0, 6, 6)$ is a solution for the CSP (11) given in Example 7. For the c-representation $\kappa_{\vec{\eta}_4}$ given in Table 1 it holds that

$$\kappa_{\vec{\eta}_4}(b \wedge (c \vee s) \wedge t) = 4 > 3 = \kappa_{\vec{\eta}_4}(b \wedge (c \vee s) \wedge \overline{t})$$

Therefore, $b \wedge (c \vee s) \mid\!\sim_{\kappa_{\vec{\eta}_4}} \overline{t}$, which implies (23). □

The situation with respect to (RM) does not change for weakly skeptical inference when we move to minimal c-representations.

Proposition 16. *Weakly skeptical inference over any of the introduced sets of minimal c-representations does not satisfy* (RM).

Proof. (Sketch) Let $\mathcal{R} = \{r_1, \ldots, r_6\}$ be the knowledge base over the signature $\Sigma = \{a, b, p, s, q\}$ with

$r_1 = (\overline{a}\overline{b}\overline{p} \vee \overline{a}b\overline{p} \vee \overline{a}bp | sq)$ $\quad r_4 = (ab \vee \overline{a}\overline{b} | psq)$
$r_2 = (a\overline{b} \vee \overline{a}\overline{b}\overline{psq} | psq \vee \overline{a}\overline{b}\overline{psq})$ $\quad r_5 = (sq | pa\overline{b} \vee p\overline{a}b)$
$r_3 = (\overline{a}b\overline{p} \vee a\overline{b}p \vee abp | sq)$ $\quad r_6 = (\overline{a}\overline{b} \vee a\overline{b}p | sq)$.

For any inference relation $\mid\!\sim^{ws,\bullet}_{\mathcal{R}}$ with $\bullet \in \{cw, +, O\}$ it holds that

$$pa\overline{b} \vee p\overline{a}b \;\mid\!\sim^{ws,\bullet}_{\mathcal{R}}\; sq \tag{24}$$
$$pa\overline{b} \vee p\overline{a}b \;\not\mid\!\sim^{ws,\bullet}_{\mathcal{R}}\; \overline{pa\overline{b}} \tag{25}$$
$$pa\overline{b} \;\not\mid\!\sim^{ws,\bullet}_{\mathcal{R}}\; sq \tag{26}$$

The rest of the proof showing that (24), (25) and (26) hold is along the lines of the proof of Proposition 14. □

5 Weak Rational Monotony

Requiring (RM) for a nonmonotonic inference relation is a rather strong and restrictive criterion. In the light of arguments that (RM) may be too strong, weaker notions of monotony have been proposed (Rott 2001; Gärdenfors and Makinson 1994).

Definition 17 (Weak Rational Monotony (Rott 2001))**.** *A nonmonotonic inference relation* $\mid\!\sim$ *satisfies* Weak Rational Monotony (WRM)*, if for all propositional formulas* A *and* B *it holds that*

$$\frac{\top \mid\!\sim B \qquad \top \not\mid\!\sim \overline{A}}{A \mid\!\sim B} \tag{WRM}$$

In this section we will study inference relations over sets of c-representation with respect to (WRM). First, we prove a general property of c-representations.

Proposition 18. *Let* $\mathcal{R}$ *be a knowledge base and* A *a formula. If for a c-representation* κ_1 *with* $\kappa_1 \models \mathcal{R}$ *we have* $\kappa_1(A) = 0$*, then* $\kappa(A) = 0$ *for all c-representations* κ *with* $\kappa \models \mathcal{R}$.

Proof. There must be a world ω with $\omega \models A$ and $\kappa_1(\omega) = 0$. To prove that ω does not falsify any conditional in $\mathcal{R}$, let us assume that ω falsifies $(B|A) \in \mathcal{R}$. Since $\omega \models A\overline{B}$ and $\kappa_1 \models (B|A)$, we get $\kappa_1(AB) < \kappa_1(A\overline{B}) \leqslant \kappa_1(\omega) = 0$, a contradiction since ranks are not negative. Thus, ω does not falsify any conditional in $\mathcal{R}$, and according to (3), $\kappa(A) = \kappa(\omega) = 0$ for all c-representations κ with $\kappa \models \mathcal{R}$. □

Proposition 19. *Skeptical inference relations defined over any set* M *of c-representations satisfy* (WRM).

Proof. Let $\mathcal{R}$ be a knowledge base and M be any set of c-representations accepting $\mathcal{R}$. Let A and B be formulas with

$$\top \;\mid\!\sim^{sk,M}_{\mathcal{R}}\; B \tag{27}$$
$$\top \;\not\mid\!\sim^{sk,M}_{\mathcal{R}}\; \overline{A}. \tag{28}$$

From (2) and (27) we conclude that for all c-representations κ in M it holds that

$$\kappa(B) = 0 < \kappa(\overline{B}). \tag{29}$$

Assume that $A \;\not\mid\!\sim^{sk,M}_{\mathcal{R}}\; B$. Then there is a $\kappa_1 \in M$ such that

$$\kappa_1(AB) \geqslant \kappa_1(A\overline{B}). \tag{30}$$

Therefore, with (29) we get that

$$\kappa_1(AB) \geqslant \kappa_1(A\overline{B}) \geqslant \kappa_1(\overline{B}) > \kappa_1(B). \tag{31}$$

From (28) we get that there is a $\kappa_2 \in M$ such that

$$\kappa_2(A) \leqslant \kappa_2(\overline{A}) \tag{32}$$

implying $\kappa_2(A) = 0$ due to (2) and hence also $\kappa_1(A) = 0$ due to Prop. 18. Using (29) and (31) we get the contradiction

$$0 = \kappa_1(A) = min\{\kappa_1(AB), \kappa_1(A\overline{B})\} > \kappa_1(B) = 0$$

and hence, $A \;\mid\!\sim^{sk,M}_{\mathcal{R}}\; B$ must hold. □

Proposition 20. *Credulous inference relations over any set of c-representations satisfy* (WRM).

Proof. Because (WRM) is strictly weaker than RM, from the proof of Proposition 12 it follows that every credulous inference relation defined over a set of c-representations satisfies (WRM). □

Proposition 21. *Weakly skeptical inference relations defined over any set M of c-representations satisfy* (WRM).

Proof. For all knowledge bases $\mathcal{R}$, every set M of c-representations accepting $\mathcal{R}$, and all formulas A, B with

$$\top \mathrel{|\!\!\sim}^{ws,M}_{\mathcal{R}} B \quad (33)$$

$$\top \mathrel{|\!\!\not\sim}^{ws,M}_{\mathcal{R}} \overline{A} \quad (34)$$

we have to show:

(I) There exists $\lambda \in M$ with $\lambda(AB) < \lambda(A\overline{B})$.

(II) For all $\kappa \in M$ we have $\kappa(AB) \leqslant \kappa(A\overline{B})$.

For both, we employ a case distinction derived form (34):

(i) For all $\kappa \in M$ we have $\kappa(\overline{A}) \geqslant \kappa(A)$, or

(ii) there exists $\kappa_1 \in M$ with $\kappa_1(\overline{A}) > \kappa_1(A)$.

I(i): Due to (2) and (33), there is a $\kappa' \in M$ with $0 = \kappa'(B) < \kappa'(\overline{B}) \leqslant \kappa'(A\overline{B})$. From (2) and (i), we get $0 = \kappa'(A) \leqslant \kappa'(\overline{A})$ and thus $0 = \kappa'(AB) < \kappa'(\overline{B}) \leqslant \kappa'(A\overline{B})$. Hence, choosing $\lambda = k'$ proves (I).

I(ii): Due to (2), we get $\kappa_1(\overline{A}) > \kappa_1(A) = 0 = min\{\kappa_1(AB), \kappa_1(A\overline{B})\}$. Because $\kappa_1'(A\overline{B}) = 0$ implies $\kappa_1'(\overline{B}) = 0$, due to Prop. 18 it also implies $\kappa(\overline{B}) = 0$ for all c-representations κ accepting $\mathcal{R}$, contradicting (33). Therefore, $\kappa_1'(AB) < \kappa_1'(A\overline{B})$ must hold, ensuring (I).

II(i): For all $\kappa \in M$, from (2) and (i) we get $0 = \kappa(A) = min\{\kappa(AB), \kappa(A\overline{B})\}$. If there were $\kappa_1 \in M$ with $\kappa_1(AB) > \kappa_1(A\overline{B})$, in contrast to (II), we would thus get $\kappa_1(A\overline{B}) = 0$ and hence $\kappa_1(\overline{B}) = 0$. Due to Prop. 18, this implies $\kappa(\overline{B}) = 0$ for all c-representations κ accepting $\mathcal{R}$, contradicting (33).

II(ii): Due to (2) and (33), we get $\kappa_1(\overline{B}) \geqslant \kappa_1(B) = min\{\kappa_1(AB), \kappa_1(\overline{A}B)\} = 0$. Because $\kappa_1(\overline{A}B) = 0$ implies $\kappa_1(\overline{A}) = 0$, contradicting (ii), $\kappa_1(AB) = 0$ must hold. Due to Prop. 18, we get $\kappa(AB) = 0 \leqslant \kappa(A\overline{B})$ for all c-representations κ accepting $\mathcal{R}$, ensuring (II). □

Since Proposition 19, 20, and 21 hold for any set M of c-representations of a knowledge base $\mathcal{R}$, they imply that skeptical, credulous, and weakly skeptical c-inference (over all c-representations) as well as the corresponding inference relations over minimal c-representations ($\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$, $\mathrel{|\!\!\sim}^{cr,\bullet}_{\mathcal{R}}$ and $\mathrel{|\!\!\sim}^{ws,\bullet}_{\mathcal{R}}$ with $\bullet \in \{cw, +, O\}$) all satisfy (WRM).

6 Conclusions and Future Work

For inference relations induced by sets of minimal c-representations, we investigated whether they satisfy the postulates of rational monotony (RM) and weak rational monotony (WRM). For skeptical, weakly skeptical, and credulous c-inference over sets of minimal c-representations we presented solutions to these previously open questions. Furthermore, we proved that every of these three inference modes over any set of c-representations satisfies (WRM).

While several postulates other than (RM) and (WRM) have already been taken into account with respect to c-inference relations (e.g. in (Beierle et al. 2016; 2018; Thorn et al. 2015)), in future work we will study c-inference with respect to further postulates that have been suggested for nonmonotonic inference relations.

References

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2016. Skeptical, weakly skeptical, and credulous inference based on preferred ranking functions. In *ECAI-2016*, volume 285, 1149–1157. IOS Press.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2018. Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem. *Ann. Math. Artif. Intell.* 83(3-4):247–275.

Beierle, C.; Kutsch, S.; and Sauerwald, K. 2018. Compilation of conditional knowledge bases for computing c-inference relations. In Ferrarotti, F., and Woltran, S., eds., *Foundations of Information and Knowledge Systems - 10th International Symposium, FoIKS 2018, Budapest, Hungary, May 14-18, 2018, Proceedings*, volume 10833 of *Lecture Notes in Computer Science*, 34–54. Springer.

Benferhat, S.; Dubois, D.; and Prade, H. 1999. Possibilistic and standard probabilistic semantics of conditional knowledge bases. *J. of Logic and Computation* 9(6):873–895.

de Finetti, B. 1937. La prévision, ses lois logiques et ses sources subjectives. *Ann. Inst. H. Poincaré* 7(1):1–68. Engl. transl. *Theory of Probability*, J. Wiley & Sons, 1974.

Dubois, D., and Prade, H. 1994. Conditional objects as nonmonotonic consequence relations: Main results. In *Proc. KR'94*, 170–177. Morgan Kaufmann Publishers.

Gärdenfors, P., and Makinson, D. 1994. Nonmonotonic inference based on expectations. *Artificial Intelligence* 65:197–245.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence* 84:57–112.

Kern-Isberner, G. 2001. *Conditionals in nonmonotonic reasoning and belief revision*, volume 2087 of *LNAI*. Springer.

Kern-Isberner, G. 2004. A thorough axiomatization of a principle of conditional preservation in belief revision. *Ann. of Math. and Artif. Intell.* 40(1-2):127–164.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artif. Intell.* 55:1–60.

Lewis, D. 1973. *Counterfactuals*. Cambridge, Mass.: Harvard University Press.

Pearl, J. 1990. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proc. TARK'1990*, 121–135. Morgan Kaufmann Publ. Inc.

Rott, H. 2001. *Change, choice and inference: A study of belief revision and nonmonotonic reasoning*. Oxford University Press.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford, UK: Oxford University Press.

Thorn, P. D.; Eichhorn, C.; Kern-Isberner, G.; and Schurz, G. 2015. Qualitative probabilistic inference with default inheritance. In *DKB/KIK-2015*, volume 1444 of *CEUR Workshop Proceedings*, 16–28. CEUR-WS.org.