Semantic Web and Ignorance: Dempster-Shafer Description Logics

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Abstract

Information incompleteness, or ignorance, is an issue that we have to consider in Semantic Web applications. Dempster-Shafer theory has been traditionally applied in information incompleteness situations. On the other hand, logic plays a major role in the Semantic Web community. In this paper, we propose a framework that applies Dempster-Shafer theory in a Description Logic Knowledge Base environment. We name our model a *Dempster-Shafer DL Knowledge Base*.

Introduction

In developing Semantic Web applications, we often come across information incompleteness issues. As an example, let us consider a data source that contains information about *hotels*. We assume each hotel h to be assigned an interval cost per night rather than a crisp value, e.g $h : [50 - 150]$. In this case, if we want to make a reservation, we do not know exactly what the cost is but we know a lower-upper bound of the cost value. Moreover, if consider the query: *I'm looking for a hotel with cost no greater than 100*, then in a crisp logic framework, where each hotel has a unique value cost, the query could be answered with a yes/no statement. In our case, where we have to deal with interval value form, a yes/no statement cannot fully answer this query. The introduction of a *degree* notion seems to be more suitable to describe this kind of information. In a Description Logics environment, if we consider a concept *DesiredHotel*, defined as $DesiredHotel = Hotel \sqcap \exists cost. \leq_{100}$, then, the answer to our query is to decide whether a hotel individual is a member of the concept *DesiredHotel*.

Information incompleteness can be classified as an uncertainty problem, other uncertainty problems consider information randomness and data inconsistency (Dubois 2007). The Dempster-Shafer theory, along with Dempster's rule of combination (Sentz and Ferson 2002), is a framework for dealing with information incompleteness, allowing integration of information from different independent sources. In this paper, we propose an adaptation of Dempster-Shafer theory in a logic context. In our work, we define an extension of crisp Knowledge Bases with Dempster-Shafer modules. The concept of *Dempster-Shafer DL Knowledge Base* is introduced and it is served as a way to tackle information incompleteness. Dempster-Shafer Theory is more well-suited in modelling beliefs regarding the truthness of an event. A framework that employs Dempster-Shafer theory is described in (Karanikola, Karali, and McClean 2013).

The rest of this paper is organized as follows: In Section 2, the basics of Description Logics are introduced. In Section 3, an overview of Dempster-Shafer Theory is presented along with some frameworks of its application in Logic. In Section 4, our method, i.e the *Dempster-Shafer DL Knowledge Base*, is defined. In Section 5, decidability and reasoning issues are considered. Finally, we give our Conclusion, where future work is also outlined.

Description Logics Overview and Uncertainty Extensions - The DL $\mathcal{ALC}$

In this Section, we overview the basics of Description Logics and present the DL $\mathcal{ALC}$. Description Logics (DL for short) are a family of knowledge representation languages. Some papers that introduce DLs are (Krötzsch, Simancik, and Horrocks 2012), (Baader, Horrocks, and Sattler 2008), (Franz, Ian, and Ulrike 2005). Generally, Description Logics define a DL Knowledge Base as a triple $< \mathcal{T}, \mathcal{R}, \mathcal{A} >$, where $\mathcal{T}$, the $TBox$ of the Knowledge Base, contains axioms concerning *DL Concepts*, $\mathcal{R}$, the $RBox$ of the Knowledge Base, contains axioms concerning *DL Roles* and $\mathcal{A}$, the $ABox$ of the Knowledge Base, contains axioms concerning *DL individuals*. A DL *interpretation* $\mathcal{I}$ is defined as $< \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} >$, where $\Delta^{\mathcal{I}}$ is the interpretation domain and $\cdot^{\mathcal{I}}$ is the interpretation function. An interpretation actually assigns a true/false value to each DL axiom. In case an axiom τ *is true wrt.* $\mathcal{I}$, this is denoted as $\mathcal{I} \models \tau$. *Subsumption*, *instantiation* and *consistency checking* are the main forms of reasoning in DLs.

$\mathcal{ALC}$ is considered the basic DL language. Its syntax uses the following sets: N_C (the set of concept names), N_R (the set of role names) and N_I (the set of individuals). In order to build complex rules, we apply a set of syntax rules. More precisely, $\mathcal{ALC}$ *concepts* are the following:

- $\top, \bot, A$, where A is a primitive concept
- If C, D are $\mathcal{ALC}$ *concepts*, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$ and $\exists r.C$, where r is a *DL Role*, are $\mathcal{ALC}$ *concepts*.

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ performs the following mapping:

$$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}, \bot^{\mathcal{I}} = \emptyset, C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}, r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$$
$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$$
$$(\forall r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \forall d', (d, d') \in r^{\mathcal{I}} \textit{ implies } d' \in C^{\mathcal{I}}\}$$
$$(\exists r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \exists d', (d, d') \in r^{\mathcal{I}} \textit{ and } d' \in C^{\mathcal{I}}\}$$

In addition, $\mathcal{ALC}$ considers two kinds of assertions for an individual α: $C(\alpha)$, meaning that α is an instance of C (*concept assertion*) and $r(\alpha, \beta)$, meaning that there is a relation r between α, β (*role assertion*). A set of concept assertions $\{C(\alpha_1), \ldots, C(\alpha_n)\}$ is *satisfied* in an interpretation $\mathcal{I}$, iff $\alpha_i^{\mathcal{I}} \in C^{\mathcal{I}}, i = 1, \ldots, n$. A set of role assertions $\{r(\alpha_1, \beta_1), \ldots, r(\alpha_n, \beta_n)\}$ is *satisfied* in an interpretation $\mathcal{I}$, iff $(\alpha_i^{\mathcal{I}}, \beta_i^{\mathcal{I}}) \in r^{\mathcal{I}}, i = 1, \ldots, n$.

There exist various frameworks for representing uncertainty in DLs. These approaches can be classified either as probabilistic or possibilistic, based on the logic behind them. For an overview on Possibility or Probability theory see (Dubois and Prade 2001). In (Lukasiewicz 2008), a framework for representing and reasoning over probabilistic uncertainty in a DL environment is outlined. The syntax is defined as a language of *conditional constraints* as an expression of the form $(\psi|\phi)[l, u]$, where ψ, ϕ are concepts and l, u are real numbers in $[0, 1]$. A possibilistic DL, which is based on a possibilistic interpretation, is described in (Qi, Pan, and Ji 2007). In this approach, possibilistic axioms are defined as (ϕ, α), where ϕ is crisp DL axiom and $\alpha \in (0, 1]$.

Dempster-Shafer Theory

Dempster-Shafer theory has been evolved as a method for representing incomplete information (Shafer 1976), (Liu and Yager 2008), (Dubois and Prade 2008). Generally, Dempster-Shafer can be considered a model of subjective probabilities. This theory employs the concept of the frame of discernment $\mathcal{W}$, which is defined as a set of exhaustive and mutually exclusive events. Then, the power set of $\mathcal{W}$, denoted as $2^{\mathcal{W}}$, is defined as the set of all subsets of $\mathcal{W}$. On each element of the set $2^{\mathcal{W}}$, the *basic probability assignment* or *mass function*, denoted as m, is defined as $m : 2^{\mathcal{W}} \to [0, 1]$, where $\sum_{A \in 2^{\mathcal{W}}} m(A) = 1$ and $m(\emptyset) = 0$.

Basic probability assignment is used in order to define *belief* and *plausibility* functions, on a subset $B \subseteq \mathcal{W}$, which constitute lower and upper probability measures:

$$Bel(B) = \sum_{A \subseteq B} m(A), \quad Pl(B) = \sum_{A \bigcap B \neq \emptyset} m(A)$$

Dempster's rule of Combination (Sentz and Ferson 2002) is defined on two basic probability assignments m_1, m_2, derived from independent sources:

$$m_1 \bigoplus m_2(B) = \frac{\sum_{A_i \bigcap A_j = B} m_1(A_i) \times m_2(A_j)}{1 - \sum_{A_i \bigcap A_j = \emptyset} m_1(A_i) \times m_2(A_j)}$$

Unifying Dempster Shafer and Logic or relate Dempster Shafer with Logic have been studied in a variety of works (Provan 1990; 1989; Saffiotti 1992; Pearl 1990; Zhu and Lee 1993). An approach for relating Belief Functions and Logic is described in (Saffiotti 1992). In this work, *bf-formulas* are considered, which are formulas of the form $F : [a, b]$, where F is a classical first-order sentence and $[a, b]$ constitutes a Belief-Plausibility interval, in a Dempster-Shafer framework. From semantics point of view, a set of classical first-order interpretations $\mathcal{I}$ is considered, and a *bf-interpretation* is defined as $\mathcal{M} : 2^{\mathcal{I}} \to [0, 1]$. that can be used to define a Belief function.

Our approach: Dempster-Shafer DL Knowledge Bases

In this Section, we present our approach which introduces the concept of the *Dempster-Shafer DL Knowledge Base*, based on the logic defined in (Saffiotti 1992). In order to do this, we extend classical DL axioms with Belief degree conditions and Plausibility degree conditions. Then, we interpret these axioms to hold with a Belief degree lower bound or Plausibility degree lower bound.

Following, we define our $\mathcal{DS} - \mathcal{ALC}$ syntax and semantics.

Syntax of $\mathcal{DS} - \mathcal{ALC}$

A *Dempster-Shafer DL knowledge base* is described by the following:

- A set $\Phi = \{p_1, p_2, \ldots, p_n\}$, where p_i is a basic crisp DL $\mathcal{ALC}$ assertional axiom.
- Any assertion ϕ is an atomic assertion, or a boolean combination of assertions.
- A set of constraints:
  - *Belief Constraints*: They have the form $\phi \quad B \geq \alpha$, and interpreted as ϕ *is true with Belief degree at least* α.
  - *Plausibility Constraints*: They have the form $\phi \quad P \geq \alpha$, and interpreted as ϕ *is true with Plausibility degree at least* α.

Definition 1. A Dempster-Shafer DL Knowledge Base is defined as a set of Belief Constraints $\mathcal{B}$ and a set of Plausibility Constraints $\mathcal{P}$, denoted as $\mathcal{KB} = (\mathcal{B}, \mathcal{P})$.

Semantics of $\mathcal{DS} - \mathcal{ALC}$

Before defining the semantics of our framework, we introduce the concept of a *possible world* I to be a subset of the set of basic crisp DL assertions Φ. In that sense, a possible world I specifies the set of assertions that are *true* in that world. We denote as $\mathcal{W}$ the set of possible worlds I, i.e $\mathcal{W} = 2^{\Phi}$. Since Φ is finite, $\mathcal{W}$ is also finite. Given a crisp DL Knowledge Base, $\mathcal{KB}_{crisp}$, and a possible world I, the satisfaction of $\mathcal{KB}_{crisp}$ is defined as:

Definition 2. A possible world I *satisfies* (or it is a *model* of) $\mathcal{KB}_{crisp}$ iff $\{p_i \mid p_i \in I\} \cup \mathcal{KB}_{crisp}$ is satisfiable.

Next, we will prove that the satisfaction (entailment) of a $\mathcal{KB}_{crisp}$ is a necessary and sufficient condition for the existence of a model $\mathcal{I}$ of this Knowledge Base. Our proof is based on the one defined in (Lukasiewicz 2008), adapted in our DL.

Proposition 1. Let Φ be a finite set of DL assertions and let $\mathcal{KB}_{crisp}$ be a crisp $\mathcal{ALC}$ Knowledge Base out of Φ. Then $\mathcal{KB}_{crisp}$ has a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ iff there exists a possible world I that satisfies $\mathcal{KB}_{crisp}$.

Proof ($\Rightarrow$)Suppose that $\mathcal{KB}_{crisp}$ has a model. This means that an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ exists. Then, the set of DL assertions that are satisfiable under $\mathcal{I}$ constitutes a subset of Φ, i.e a possible world I. This means that $\mathcal{KB}_{crisp}$ has also a model I. ($\Leftarrow$)Suppose that there exists a I model of $\mathcal{KB}_{crisp}$. This means that $\mathcal{KB}_{crisp}$ is satisfiable, so a model $\mathcal{I}$ exists. $\square$

The set of possible worlds $\mathcal{W}$ can be considered as a Dempster-Shafer frame of discernment, since the elements of $\mathcal{W}$ are mutually exclusive. We define a *Dempster-Shafer interpretation* m, as a basic probability assignment function on subsets of the set $\mathcal{W}$. Based on this assignment, we define belief and plausibility degrees, induced from bpa's on sets $T \subseteq \mathcal{W}$. In addition, the power set of $\mathcal{W}$, denoted as $\mathcal{PW}$ is defined over the following function:

$$\mathcal{PW} = 2^{\mathcal{W}}$$

In this context, we consider that a crisp $\mathcal{ALC}$ axiom can be true in a subset of $\mathcal{W}$. We define this subset as a set-interpretation, i.e:

Definition 3. Let us consider the set of all possible worlds (or interpretations) $\mathcal{W}$, with power-set $2^{\mathcal{W}}$. Any $K \in 2^{\mathcal{W}}$ is called a *set-interpretation*.

Our set-interpretation is defined in an analogous way to a hyper-interpretation (Saffiotti 1992).

The entailment of an axiom ϕ under a set-interpretation is defined as:

Definition 4. An entailment of an axiom ϕ from a set-interpretation K, where $K \in 2^{\mathcal{I}}$ is defined as:

$$K \models \phi \text{ iff }, \forall\, I \in\ K,\ I \models_{DL} \phi$$
$$K \not\models \phi \text{ iff }, \exists\, I \in\ K,\ I \not\models_{DL} \phi$$
$$K \models \neg\phi \text{ iff }, \forall\, I \in\ K,\ I \not\models_{DL} \phi$$

In the definition above, $\models_{DL}$ denotes classical crisp DL entailment.

Definition 5. A *Dempster-Shafer* interpretation m is defined as a basic probability assignment, as $m : 2^{\mathcal{W}} \rightarrow [0, 1]$.

As we operate on a Dempster-Shafer framework, a constraint that we have to preserve is the following:

$$\sum_{T \in\, 2^{\mathcal{W}}} m(T) = 1$$

Our Dempster-Shafer DL knowledge base assumes a set of possible worlds $\mathcal{W}$ and assigns a *Dempster-Shafer interpretation* to subsets of this set. Any $A \subseteq \mathcal{W}$ such that $m(A) > 0$ constitutes a *focal set-interpretation*. Following, we define Belief and Plausibility Degrees of assertions ϕ from these focal interpretations, based on *entailment* notion related to $\mathcal{PW}$.

Definition 6. The Belief Degree of an axiom ϕ under a Dempster-Shafer interpretation m is defined as:

$$Bel_m(\phi) = \sum_{PW \models \phi} m(PW), PW \in \mathcal{PW}$$

Definition 7. The Belief Degree of an axiom $\neg\phi$ under a Dempster-Shafer interpretation m is defined as:

$$Bel_m(\neg\phi) = \sum_{PW \not\models \phi} m(PW), PW \in \mathcal{PW}$$

In a Dempster-Shafer framework, the following relation holds for an axiom ϕ:

$$Pl_m(\phi) = 1 - Bel_m(\neg\phi)$$

Proposition 2. Based on the previous definition and according to Belief - Plausibility relation, we have that a Plausibility Degree for an axiom ϕ is equal to:

$$Pl_m(\phi) = 1 - \sum_{PW \not\models \phi} m(PW), PW \in \mathcal{PW}$$

Definition 8. The truthness of a Dempster-Shafer axiom ϕ under a Dempster-Shafer interpretation m is defined as:

$$m \models \phi\, \mathcal{B} \geq \alpha \ \text{ iff }\ Bel(\phi) \geq \alpha$$
$$m \models \phi\, \mathcal{P} \geq \alpha \ \text{ iff }\ Pl(\phi) \geq \alpha$$

A Dempster-Shafer interpretation m is a *model* of a Dempster-Shafer DL Knowledge Base $\mathcal{KB} = (\mathcal{B}, \mathcal{P})$ iff $m \models \mathcal{U}, \quad \forall \mathcal{U} \in \mathcal{B} \cup \mathcal{P}$. A Dempster-Shafer axiom ϕ is a *logical consequence* of a Dempster-Shafer DL Knowledge Base $\mathcal{KB}$, denoted as $\mathcal{KB} \models \phi$, iff every model of $\mathcal{KB}$ is also a model of ϕ. A Dempster-Shafer DL Knowledge Base is *consistent* if a model exists for $\mathcal{KB}$.

Finally, in an analogous way defined in (Straccia 1998), since a crisp DL interpretation can be considered as a Dempster-Shafer interpretation (with constraints of value 1.0), we have the following proposition:

Proposition 3. Let $\mathcal{KB}$ a Dempster-Shafer DL Knowledge Base. Then, we define as the crisp counterpart of $\mathcal{KB}$ the crisp Knowledge Base $\overline{\mathcal{KB}} = \{a : a \quad \mathcal{B} \geq n \mid a \quad \mathcal{P} \geq m\}$. If $\mathcal{KB} \models a \quad \mathcal{B} \geq n$ (or $\mathcal{KB} \models a \quad \mathcal{P} \geq m$), then $\overline{\mathcal{KB}} \models_{crisp} a$.

In addition, as in (Straccia 1998), we define as $\widehat{\mathcal{KB}} = \{a \quad \mathcal{B} \geq 1\} \cup \{a \quad \mathcal{P} \geq 1\}, a \in \mathcal{KB}_c$, where $\mathcal{KB}_c$ is a crisp Knowledge Base. Then, the following holds:

$$\textit{If } \mathcal{KB}_c \models_{crisp} a \textit{ then } \widehat{\mathcal{KB}} \models a \quad \mathcal{B} \geq 1 \textit{ and}$$
$$\widehat{\mathcal{KB}} \models a \quad \mathcal{P} \geq 1$$

For a detailed proof of the proposition above, see (Straccia 1998).

An important issue considers consistency checking. Knowledge Base consistency partially stems from the relation between Belief and Plausibility degrees, i.e:

$$Bel(\phi) = 1 - Pl(\phi)$$

Consistency checking refers to the Belief - Plausibility Degrees of a formula ϕ and its negation $\neg\phi$.

Syntactic Consistency: Let us consider a Dempster-Shafer DL Knowledge Base $\mathcal{KB}$ with the following Belief and Plausibility constraints:

$$\begin{array}{ll} \phi \quad \mathcal{B} \geq \alpha, & \phi \quad \mathcal{P} \geq \beta \\ \neg\phi \quad \mathcal{B} \geq \gamma, & \neg\phi \quad \mathcal{P} \geq \delta \end{array}$$

The syntactic consistency can be proved by employing the Belief-Plausibility relation. Thus, we conclude that a consistent Knowledge Base should be aligned with the following:

$$\beta \leq (1-\gamma) \qquad \delta \leq (1-\alpha)$$

We can prove syntactic consistency, by checking whether the equations above hold.

Semantic Consistency: From a semantics point of view, if m a model of $\mathcal{KB}$, i.e:

$$\begin{array}{ll} m \models \phi \quad \mathcal{B} \geq \alpha, & m \models \phi \quad \mathcal{P} \geq \beta \\ m \models \neg\phi \quad \mathcal{B} \geq \gamma, & m \models \neg\phi \quad \mathcal{P} \geq \delta \end{array}$$

Then, following our semantics definition, we conclude that:

$$\begin{array}{ll} Bel(\phi) \geq \alpha, & Bel(\phi) \geq \beta, \\ Pl(\neg\phi) \geq \gamma, & Pl(\neg\phi) \geq \delta \end{array}$$

For $\mathcal{KB}$ being a consistent Knowledge Base, we have always to preserve that $Bel(\phi) + Pl(\neg\phi) = 1$.

Example In order to illustrate our method, let us consider the following Dempster-Shafer DL Knowledge Base:

$$\begin{array}{ll} < Hotel \sqcap \exists cost. \leq_{100} (h_1) > & \mathcal{B} \geq 0.5 \\ < Hotel \sqcap \exists cost. \leq_{100} (h_1) > & \mathcal{P} \geq 0.7 \end{array}$$

Our knowledge base is consistent, based on consistency checking formulas, defined in the previous section. Now, let us suppose that we add the following axiom:

$$\neg < Hotel \sqcap \exists cost. \leq_{100} (h_1) > \quad \mathcal{B} \geq 0.9$$

Based on the consistency checking, we must have $0.7 \leq 0.1$, which makes our knowledge base inconsistent.

Combined Dempster-Shafer entailment

In this Section, a new notion of entailment, named *Combined Dempster-Shafer entailment* and denoted as $\models_{DS_{combined}}$ is defined. The Combined Dempster-Shafer entailment is applied on two different independent Knowledge Bases, named $\mathcal{KB}_1 = (\mathcal{B}_1, \mathcal{P}_1)$ and $\mathcal{KB}_2 = (\mathcal{B}_2, \mathcal{P}_2)$ and combine assertions that are entailed (with a Belief-Plausibility degree) by both Knowledge Bases. The intuition behind this entailment resides in the fact that we can derive the truthness of a statement, based on different sources (KBs) that entail this statement.

Let us suppose that the following hold:

$$\mathcal{KB}_1 \models \phi : \mathcal{B} \geq \gamma \;\; , \mathcal{KB}_2 \models \phi : \mathcal{B} \geq \delta$$

This means that, if m_1 is a model of $\mathcal{KB}_1$ and m_2 is a model of $\mathcal{KB}_2$, then we have the following:

$$Bel_1(\phi) \geq \gamma, \;\; Bel_2(\phi) \geq \delta$$

In addition, we consider $T_i, i = 1, \ldots, n$ the focal set-interpretations of $\mathcal{KB}_1$ and $T_j, j = 1, \ldots, m$, the focal set-interpretations of $\mathcal{KB}_2$.

In our framework, Dempster's rule of Combination is applied in order to define a *Combined Belief Degree*. This rule provides for combination of a set of bpa's $m_1, \ldots, m_n$. In our approach, the Belief Degree of ϕ equals its bpa value, as subsets of ϕ are not feasible to be defined, i.e. $Bel(\phi) = m(\phi)$. Belief and Plausibility degrees can be defined as a combination of m_1, m_2. We apply our Combined Belief Degree based on Def.6 and by considering intersections of the form $T_i \cap T_j$ just like in Dempster's rule of combination. Then, we introduce the following definition:

Definition 9. The Combined Belief Degree $Bel_{1,2}$ over models m_1 and m_2, is defined as:

$$Bel_{1,2}(\phi) = \frac{\sum_{T_i \cap T_j \models \phi} m_1(T_i) \times m_2(T_j)}{1 - \sum_{T_i \cap T_j = \emptyset} m_1(T_i) \times m_2(T_j)}$$

The Combined Plausibility Degree $Pl_{1,2}$ over models m_1 and m_2, is derived from the following formula:

$$Pl_{1,2}(\phi) = 1 - Bel_{combined}(\neg\phi)$$

Let us consider $\mathcal{KB}_1$ with model m_1 and $\mathcal{KB}_2$ with model m_2. Then, the Dempster-Shafer Combined entailment is defined as follows:

Definition 10. An axiom $\phi \quad \mathcal{B} \geq \varepsilon$ is Dempster-Shafer Combined entailed, under $\mathcal{KB}_1$ and $\mathcal{KB}_2$, denoted as $\mathcal{KB}_1 \oplus \mathcal{KB}_2 \models_{DS_{combined}} \phi \quad \mathcal{B} \geq \varepsilon$, iff $\varepsilon \geq Bel_{1,2}(\phi)$.

Definition 11. An axiom $\phi \quad \mathcal{P} \geq \varepsilon$ is Dempster-Shafer Combined entailed, under $\mathcal{KB}_1$ and $\mathcal{KB}_2$, denoted as $\mathcal{KB}_1 \oplus \mathcal{KB}_2 \models_{DS_{combined}} \phi \quad \mathcal{P} \geq \varepsilon$, iff $\varepsilon \geq Pl_{1,2}(\phi)$.

Example Continuing the previous example, let us suppose, that we have two Dempster-Shafer DL Knowledge Bases, $\mathcal{KB}_1$ and $\mathcal{KB}_2$ consisting of the following axioms:

$$\begin{array}{ll} \mathcal{KB}_1 : < Hotel \sqcap \exists cost. \leq_{100} (h_1) > & \mathcal{B} \geq 0.5 \\ \mathcal{KB}_2 : < Hotel \sqcap \exists cost. \leq_{100} (h_1) > & \mathcal{B} \geq 0.7 \end{array}$$

We consider $\mathcal{W} = \{I_1, I_2\}$, two possible worlds, where $< Hotel \sqcap \exists cost. \leq_{100} (h_1) >$ is false in I_1 and true in I_2, i.e, there exist two DL interpretations, $\mathcal{I}_1$ and $\mathcal{I}_2$ such that:

$$\begin{array}{l} \mathcal{I}_1 \not\models < Hotel \sqcap \exists cost. \leq_{100} (h_1) > \\ \mathcal{I}_2 \models < Hotel \sqcap \exists cost. \leq_{100} (h_1) > \end{array}$$

Table 1: Dempster-Shafer Interpretation

$\mathcal{PW}$	$\models \phi$	m_1	m_2
$\{I_1\}$	0	0.3	0.2
$\{I_2\}$	1	0.5	0.7
$\{I_1, I_2\}$	0	0.2	0.1

We consider two Dempster-Shafer interpretations, m_1, m_2 as described in Table 3. Also, we consider m_1 a model of $\mathcal{KB}_1$ and m_2 a model of $\mathcal{KB}_2$. For our convenience we name $< Hotel \sqcap \exists cost. \leq_{100} (h_1) >$ as ϕ.

By applying the Combination, based on our formula defined in the previous section, we derive a result of $Bel_{combined}(\phi)$ of 0.78.

Based on the Dempster-Shafer Combined entailment, the following holds:

$$\mathcal{KB}_1 \oplus \mathcal{KB}_2 \models_{DS_{combined}}$$
$$< Hotel \sqcap \exists cost. \leq_{100} (h_1) > \quad \mathcal{B} \geq 0.78$$

Decidability and Reasoning in Dempster-Shafer Description Logics

In this Section, we provide a method for reasoning over a Dempster-Shafer DL Knowledge Base, $\mathcal{KB}_{DL}$, which actually contains ABox. Reasoning in DLs is usually accomplished through tableaux procedures (Buchheit, Donini, and Schaerf 1993). The decidability problem in our framework can be reduced in finding a method for deciding whether $\mathcal{KB}_{DL} \models \tau$, where τ is a Dempster-Shafer assertion axiom. Deciding satisfiability in a Dempster-Shafer DL Knowledge Base should take into account a basic probability assignment on subsets of interpretations (or possible worlds). It has to be noted that our axioms are described by Belief and Plausibility conditions in a similar way to axioms defined in (Straccia 1998) where axioms are annotated with membership degree conditions. Having taken this into consideration, we adapt the method described in (Straccia 1998) and extend it in order to capture Belief and Plausibility conditions.

More precisely, we consider $\mathcal{O}$ as an alphabet of symbols (DL individuals) in the same way as it is referred in (Buchheit, Donini, and Schaerf 1993). Moreover, we consider an alphabet of variable symbols $\mathcal{V}$ along with an ordering $\prec$ on $\mathcal{V}$. Also, the common term *object* is employed for describing either a DL individual or a $\mathcal{V}$ variable, in other words an object is an element of $\mathcal{O} \cup \mathcal{V}$. The symbols s, t are used to denote an object element. A *constraint* σ is defined as $s : C$ or sPt, where C is a DL concept and P is a DL role. Following, a *constraint system* is defined as a finite nonempty set of constraints. Also, by $\neg\sigma$, we denote $s : \neg C$ or $s\neg Pt$.

Based on these concepts, we define a *Belief constraint* as an expression of the following forms:

$$\sigma \;\; \mathcal{B} \bowtie n$$

where $\bowtie$ is one of $<, >, \leq, \geq$.

The *Plausibility constraints* are defined in an analogous way. A *Dempster-Shafer constraint system* is defined as a set of Belief and Plausibility constraints. An interpretation m *satisfies* a Belief Constraint

$$s : C \;\; \mathcal{B} \bowtie n \qquad (sPt \;\; \mathcal{B} \bowtie n)$$

iff $Bel_m(C(s)) \bowtie n$ (Resp. $Bel_m(P(s,t)) \bowtie n$). Also, m satisfies a constraint system $\mathcal{S}$ iff m satisfies every Dempster-Shafer constraint in it.

A Dempster-Shafer DL Knowledge Base $\mathcal{KB}_{DS}$ can be mapped into a Dempster-Shafer constraint system $\mathcal{S}_{\mathcal{KB}}$, defined as:

$$\begin{aligned} \mathcal{S}_{\mathcal{KB}} = \{a : C \;\; \mathcal{B} \geq n \mid C(a) \quad \mathcal{B} \geq n\} \;\; &\cup \\ \{a : C \;\; \mathcal{P} \geq n \mid C(a) \quad \mathcal{P} \geq n\} \;\; &\cup \\ \{aPb \;\; \mathcal{B} \geq n \mid R(a,b) \quad \mathcal{B} \geq n\} \;\; &\cup \\ \{aPb \;\; \mathcal{P} \geq n \mid R(a,b) \quad \mathcal{P} \geq n\} & \end{aligned}$$

Then, we have that $\mathcal{KB}_{DS} \models C(a) \quad \mathcal{B} \geq n$ iff $\mathcal{S}_{\mathcal{KB}} \cup a : C\mathcal{B} < n$ is not satisfiable. Similarly, we operate on Plausibility conditions.

In order to examine constraint satisfiability of $\mathcal{S}_{KB}$, we consider a set of constraint propagation rules. These rules actually add constraints to $\mathcal{S}_{KB}$ until a *contradiction (or clash)* happens or the current constraint system is *complete* (i.e an m that satisfies $\mathcal{S}_{KB}$ plus a constraint to be added can be obtained from the current constraint system).

A set of Dempster-Shafer constraints $\mathcal{S}$ contains a contradiction iff it contains one of the following:

1. $\top, \bot$ contradictions:

$$\begin{gathered} s : \bot\mathcal{B} \geq n, \quad s : \bot\mathcal{P} \geq n, \quad s : \bot\mathcal{B} > n, n > 0 \\ s : \bot\mathcal{P} > n, \quad s : \bot\mathcal{B} < 0, \quad s : \bot\mathcal{P} < 0, n > 0 \\ s : \top\mathcal{B} \leq n, n < 1, \quad s : \top\mathcal{P} \leq n, n < 1, \quad s : \top\mathcal{B} < n \\ s : \top\mathcal{P} < n, \quad s : \top\mathcal{B} > 1, \quad s : \top\mathcal{P} > 1 \end{gathered}$$

2. $<, >, \leq, \geq$ relationships contradictions:

$$\begin{gathered} \sigma \;\; \mathcal{B} \geq n \;\; and \;\; \sigma \;\; \mathcal{B} < m \;\; and \;\; n \geq m \\ \sigma \;\; \mathcal{B} \geq n \;\; and \;\; \sigma \;\; \mathcal{B} \leq m \;\; and \;\; n > m \\ \sigma \;\; \mathcal{B} > n \;\; and \;\; \sigma \;\; \mathcal{B} < m \;\; and \;\; n \geq m \\ \sigma \;\; \mathcal{B} > n \;\; and \;\; \sigma \;\; \mathcal{B} \leq m \;\; and \;\; n \geq m \end{gathered}$$

In the similar way we denote plausibility contradictions.

As in (Straccia 1998), the propagation rules can have one of the following forms:

$$\Phi \rightarrow \Psi \; \textit{if} \; \Gamma \qquad \Phi \Rightarrow \Psi \; \textit{if} \; \Gamma$$

where Φ, Ψ are sequences of Dempster-Shafer constraints and Γ is a condition. A rule fires if the condition Γ holds and the current set of Dempster-Shafer constraints contains a set of constrains that match Φ. After firing the first rule deletes Φ from S and both rules add Ψ.

Since the constraints can be one of $>, <, \geq, \leq$, connectives $\sqcap, \sqcup, \neg, \forall, \exists$ and we consider two types of constraints (Belief and Plausibility), then we have a total of 40 rules. As an example, next, we will show the case for $\neg \geq$ for Plausibility constraint. In this case, we have the following rule:

$$(\neg \geq) < s : \neg C \;\; \mathcal{P} \geq n > \rightarrow < s : C \;\; \mathcal{B} \leq (1 - n) >$$

Example Let us consider the following Knowledge Base:

$$\mathcal{KB} = \{C(a) \quad \mathcal{B} \geq 0.7, \qquad \neg D(a) \quad \mathcal{P} \geq 0.9\}$$

In addition we consider the following assertions:

$$\gamma_1 : C(a) \quad \mathcal{B} \geq 0.5 \qquad \gamma_2 : \neg D(a) \quad \mathcal{P} \geq 0.8$$

We will show that $\mathcal{KB} \models \gamma_1$ and $\mathcal{KB} \models \gamma_2$.

In the first case, we can derive a *clash* for the $\mathcal{S}_{\mathcal{KB}} \cup \{a : C \ \mathcal{B} < 0.5\}$. Based on the relationships contradictions and assigning $n = 0.7$ and $m = 0.5$, we immediately derive a clash. Hence, $\mathcal{KB} \models \gamma_1$. In the second case, we can derive a *clash* for the $\mathcal{S}_{\mathcal{KB}} \cup \{a : \neg D \ \mathcal{P} < 0.8\}$. We apply the following substitutions, based on the $\neg \geq$ rule defined previously:

$$a : \neg D \ \mathcal{P} \geq 0.9 \rightarrow a : D \ \mathcal{B} \leq 0.1 \qquad (1)$$

$$a : \neg D \ \mathcal{P} < 0.8 \rightarrow a : D \ \mathcal{B} > 0.2 \qquad (2)$$

Hence, we have clash, i.e $\mathcal{KB} \models \gamma_2$.

As a final point, we consider complexity issues of our framework. In Proposition 3, we have proved the relationship between Dempster-Shafer DL satisfiability and crisp DL satisfiability. This reduces the Dempster-Shafer $\mathcal{ALC}$ satisfiability in $\mathcal{ALC}$ satisfiability, and considering that entailment problem in $\mathcal{ALC}$ is PSPACE-complete (Schmidt-Schaubßand Smolka 1991), we have that Dempster-Shafer entailment decidability is PSPACE-complete. Concerning Dempster's rule of Combination complexity in cases of Combined entailment, this according to (Orponen 1990), can be considered as $\#P$-complete problem. In addition, in (Wilson 2000) a set of algorithms is proposed in order to manage better complexity.

Conclusion

In our paper, we have defined a Dempster-Shafer DL Knowledge Base, in order to represent uncertainty in a Description Logics framework. In addition, a combination method of independent Dempster-Shafer DL Knowledge Bases has been proposed, based on Dempster's rule of Combination. Having defined our framework, we also examine decidability and reasoning issues. As a next step, focusing on reasoning methods, we shall examine cases where better complexity results can be achieved. Finally, we shall consider the extension of our Dempster-Shafer DL Knowledge Base in order to capture Fuzzy Concepts.

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