Proceedings of the Twenty-Sixth International Florida Artificial Intelligence Research Society Conference

Compiling Preference Queries in Qualitative Constraint Problems

Jean-François Condotta
Université Lille-Nord de France, Artois
CRIL-CNRS, UMR 8188
rue de l'université, SP 16, F62307 Lens
condotta@cril.fr

Souhila Kaci
Université Montpellier 2
LIRMM, UMR 5506, CC477
161 rue Ada, F34095 Montpellier
kaci@lirmm.fr

Abstract

Comparative preference statements are the basic ingredients of conditional logics for representing users' preferences in a compact way. These statements may be strict or not and obey different semantics. Algorithms have been developed in the literature to compute a preference relation over outcomes given a set of comparative preference statements and one or several semantics. These algorithms are based on insights from non-monotonic reasoning (more specifically, minimal and maximal specificity principles) enforcing the preference relations to be a complete preorder. The main limitation of these logics however relies in preference queries when comparing two outcomes. Indeed given two outcomes having the same preference w.r.t. the preference relation, there is no indication whether this equality results from an equality between two preference statements or the outcomes are in fact incomparable and equality has been enforced by specificity principles. On the other hand, comparative preference statements and their associated semantics can be translated into qualitative constraint satisfaction problems in which one can have a precise ordering over two outcomes. In this paper we investigate this bridge and provide a compilation of conditional logics-based preference queries in qualitative constraint problems.

Introduction

Preferences are useful in many real-life problems, guiding human decision making from early childhood up to complex professional and organizational decisions. It is commonly acknowledged that rank-ordering the whole set of outcomes is simply infeasible. This is because generally this set is too large. Fortunately, in practice, we have at hand preferences over partial descriptions of outcomes. These preferences may be given in different formats. However the latter implicitly or explicitly refer to comparative preference statements of the form "prefer X to Y". We may also encounter conditional comparative preference statements of the form "if Z then prefer X to Y".

Conditional logics are preference representation languages which support such preferences. They use different completion principles in order to compute a preference relation induced by a set of preference statements. They use insights from non-monotonic reasoning, namely minimal and maximal specificity principles, to compute a preference relation (which is a complete preorder) over outcomes given a set of preference statements. Conditional logics have been extended to deal with non-strict comparative preference statements. More precisely, one can state that X is at least as preferred as Y (conditionally or unconditionally). Therefore preference statements of the form "X and Y are equally preferred" can also be expressed in these logics. As preferences are expressed over partial descriptions of outcomes, each comparative preference statement leads to the comparison of two sets of outcomes. Different ways are possible to perform such a comparison. They are called preference semantics. They express more or less strong requirements on the preference relation associated to the set of comparative preference statements at hand.

In the original proposal of conditional logics, i.e. only strict preference statements are considered, two outcomes having equal preference w.r.t. the preference relation associated with a given set of preference statements is interpreted as the two outcomes are originally incomparable. Specificity principles enforce their equality because they are preferred to all outcomes with a lower rank in the preorder and less preferred to all outcomes with a higher rank w.r.t. these principles. However this assertion is no longer possible with extended conditional logics, i.e. when non-strict preference statements are present. In fact, the reason of an equality between two outcomes is lost. We do not know whether they are incomparable or equally preferred w.r.t. some non-strict preference statements. This information is important in recommendation systems in particular.

On the other hand, comparative preference statements and their associated semantics can be translated into qualitative constraint satisfaction problems in which one can have a precise ordering over two outcomes. In this paper we investigate this bridge and present a compilation of conditional logics-based preference queries in qualitative constraint problems.

After necessary background, we present conditional logics. Then we give an example to highlight the limitations of these logics. After that we provide an encoding of conditional logics and preference queries in qualitative constraint problems to solve these limitations. Lastly, we conclude.

Background

Let $V = \{V_1, \ldots, V_h\}$ be a set of h variables, each takes its values in a domain $Dom(V_i)$. A possible outcome, denoted by ω, is the result of assigning a value in $Dom(V_i)$ to each variable V_i in V. Ω denotes the set of all possible outcomes. We suppose that this set is fixed and finite. Let $\mathcal{L}$ be a language based on V. $Mod(\alpha)$ denotes the set of outcomes that make the formula α (built on $\mathcal{L}$) true. It is also called α-*outcomes*. A preference relation $\succeq$ on $\mathcal{X}= \{x, y, z, \ldots\}$ is a binary relation on $\mathcal{X} \times \mathcal{X}$ such that $x \succeq y$ stands for "x is at least as preferred as y". $x \approx y$ means that both $x \succeq y$ and $y \succeq x$ hold i.e., x and y are equally preferred. Lastly $x \sim y$ means that neither $x \succeq y$ nor $y \succeq x$ holds, i.e., x and y are incomparable. The notation $x \succ y$ means that x is strictly preferred to y. We have $x \succ y$ if $x \succeq y$ holds but $y \succeq x$ does not. $\succeq$ is a preorder on $\mathcal{X}$ iff $\succeq$ is reflexive and transitive, i.e., $\forall x \in \mathcal{X}$, $x \succeq x$ holds and $\forall x, y, z \in \mathcal{X}$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$. We suppose that a preference relation is a preorder. $\succeq$ is complete iff $\forall x, y \in \mathcal{X}$, either $x \succeq y$ or $y \succeq x$ holds. $\succeq$ is cyclic if and only if $\exists x, y \in \mathcal{X}$ such that both $x \succ y$ and $y \succ x$ hold. Otherwise it is acyclic. Given a preference relation $\succeq$ and a formula α, the set of the maximally (resp. minimally) preferred α-outcomes is denoted by $\max(\alpha, \succeq)$ (resp. $\min(\alpha, \succeq)$) and defined as $\max(\alpha, \succeq) = \{\omega | \omega \in Mod(\alpha), \nexists \omega' \in Mod(\alpha), \omega' \succ \omega\}$ (resp. $\min(\alpha, \succeq) = \{\omega | \omega \in Mod(\alpha), \nexists \omega' \in Mod(\alpha), \omega \succ \omega'\}$).

For convenience, a complete preorder $\succeq$ can also be represented by a well ordered partition of Ω. A sequence of sets of outcomes of the form $(E_1, \ldots, E_n)$ is a partition of Ω iff (i) $\forall i, E_i \neq \emptyset$, (ii) $E_1 \cup \ldots \cup E_n = \Omega$, and (iii) $\forall i, j, E_i \cap E_j = \emptyset$ for $i \neq j$. A partition of Ω is ordered if and only if it is associated with a preorder $\succeq$ on Ω such that ($\forall \omega, \omega' \in \Omega$ with $\omega \in E_i, \omega' \in E_j$ we have $i \leq j$ iff $\omega \succeq \omega'$).

Definition 1 *(Yager 1983) Let $\succeq$ and $\succeq'$ be two complete preorders on Ω represented by $(E_1, \ldots, E_n)$ and $(E'_1, \ldots, E'_{n'})$ respectively. We say that $\succeq$ is less specific than $\succeq'$, written as $\succeq \sqsubseteq \succeq'$, iff $\forall \omega \in \Omega$, if $\omega \in E_i$ and $\omega \in E'_j$ then $i \leq j$. $\succeq$ belongs to the set of minimally (resp. maximally) specific preorders, among a set of complete preorders, if and only if there is no preorder in the set that is strictly less (resp. more) specific than $\succeq$. If $\succeq$ is the unique minimally (resp. maximally) specific complete preorder then it is called the least (resp. most) specific preorder.*

Conditional logics

The basic ingredient of conditional logics are qualitative comparative preference statements of the form "prefer α to β". Handling such a preference statement is easy when both α and β refer to an outcome. However this task becomes complex when α and β refer to sets of outcomes, in particular when they share some outcomes. In order to prevent this situation von Wright (1963) interprets the statement "prefer α to β" as a choice problem between $\alpha \wedge \neg\beta$ and $\beta \wedge \neg\alpha$. Therefore the statement "prefer α to β" leads to prefer $\alpha \wedge \neg\beta$-outcomes over $\beta \wedge \neg\alpha$-outcomes. Particular situations are those when $\alpha \wedge \neg\beta$ (resp. $\beta \wedge \neg\alpha$) is a contradiction or is not feasible in which case it is replaced with α (resp. β). For simplicity we suppose that both $\alpha \wedge \neg\beta$ and $\beta \wedge \neg\alpha$ are consistent and feasible.

Let us mention that comparative preference statements may be expressed w.r.t. some context. They are of the form "if γ, prefer α to β" which stands for "prefer α to β when γ is true". This also means that we compare $\gamma \wedge \alpha \wedge \neg\beta$-outcomes and $\gamma \wedge \beta \wedge \neg\alpha$-outcomes which corresponds to "prefer $\gamma \wedge \alpha$ to $\gamma \wedge \beta$". Indeed without loss of generality, we focus on statements of the form "prefer α to β".

Lastly, conditional logics have been extended to deal with non-strict comparative preference statements of the form "α is at least as preferred as β". Whatever the preference of α over β being strict or not, we have to compare $\alpha \wedge \neg\beta$-outcomes and $\beta \wedge \neg\alpha$-outcomes.

Different ways were proposed in the literature to compare two sets of objects. They are called preference semantics.

Preference semantics

We denote by $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$) a comparative preference statement "prefer α to β" (α is at least as preferred as β). A preference semantics refers to the way $\alpha \wedge \neg\beta$-outcomes and $\beta \wedge \neg\alpha$-outcomes are rank-ordered w.r.t. a preference relation $\succeq$ (to be constructed). More precisely, it expresses a constraint on $\succeq$ in order to get $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$) satisfied by $\succeq$. Different ways have been studied for comparing two sets of objects leading to different preference semantics. We recall the most used semantics (Boutilier 1994; Benferhat et al. 2002; van der Torre and Weydert 2001).

Definition 2 *Let $\succeq$ be a preference relation. Consider $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$).*

- **Strong semantics** *$\succeq$ satisfies $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$), denoted by $\succeq \models_{st} \alpha \rhd \beta$ (resp. $\succeq \models_{st} \alpha \unrhd \beta$), iff $\forall \omega \in \min(\alpha \wedge \neg\beta, \succeq)$, $\forall \omega' \in \max(\beta \wedge \neg\alpha, \succeq)$, $\omega \succ \omega'$ (resp. $\omega \succeq \omega'$).*
- **Optimistic semantics** *$\succeq$ satisfies $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$), denoted by $\succeq \models_{opt} \alpha \rhd \beta$ (resp. $\succeq \models_{opt} \alpha \unrhd \beta$), iff $\forall \omega \in \max(\alpha \wedge \neg\beta, \succeq)$, $\forall \omega' \in \max(\beta \wedge \neg\alpha, \succeq)$, $\omega \succ \omega'$ (resp. $\omega \succeq \omega'$).*
- **Pessimistic semantics** *$\succeq$ satisfies $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$), denoted by $\succeq \models_{pes} \alpha \rhd \beta$ (resp. $\succeq \models_{pes} \alpha \unrhd \beta$), iff $\forall \omega \in \min(\alpha \wedge \neg\beta, \succeq)$, $\forall \omega' \in \min(\beta \wedge \neg\alpha, \succeq)$, $\omega \succ \omega'$ (resp. $\omega \succeq \omega'$).*
- **Opportunistic semantics** *$\succeq$ satisfies $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$), denoted by $\succeq \models_{opp} \alpha \rhd \beta$ (resp. $\succeq \models_{opp} \alpha \unrhd \beta$), iff $\forall \omega \in \max(\alpha \wedge \neg\beta, \succeq)$, $\forall \omega' \in \min(\beta \wedge \neg\alpha, \succeq)$, $\omega \succ \omega'$ (resp. $\omega \succeq \omega'$).*

We shall abuse notation and write $\alpha \rhd_* \beta$ (resp. $\alpha \unrhd_* \beta$), with $* \in \{st, opt, pes, opp\}$, to say that $\alpha \rhd \beta$ (resp. $\alpha \unrhd \beta$) is interpreted following the corresponding semantics, namely $*$. Therefore we say that $\succeq$ satisfies $\alpha \rhd_* \beta$ (resp. $\alpha \unrhd_* \beta$) to mean that $\succeq \models_* \alpha \rhd \beta$ (resp. $\succeq \models_* \alpha \unrhd \beta$).

Definition 3 *A preference set of type $*$, denoted by $\mathcal{P}_*$, is a set of preferences of the form $\{p_i \rhd_* q_i, p_i \unrhd_* q_i | i = 1, \ldots, n\}$, with $* \in \{st, opt, pes, opp\}$. An acyclic preference relation $\succeq$ is a model of $\mathcal{P}_*$ if and only if $\succeq$ satisfies each preference $p_i \rhd_* q_i$ (resp. $p_i \unrhd_* q_i$) in $\mathcal{P}_*$. A preference set $\mathcal{P}_*$ is consistent if it has a model.*

From preference sets to preference relations

Generally we have to deal with several comparative preference statements expressed by an individual. There are mainly two kinds of queries in preference representation: either one looks for the maximally preferred outcomes or compares two outcomes. In many applications (for e.g. database queries), individuals are more concerned with the preferred outcomes. In the case where these outcomes are not satisfactory (e.g. preferred menus are too expensive), then we need to compute the preferred outcomes among remaining ones, and so on. In order to accommodate these considerations, we associate a complete preorder to a preference set. Different complete preorders may satisfy (i.e., are models of) a preference set given a semantics. However it is widely acknowledged that, for decision purposes, it is more convenient to characterize a *unique* complete preorder (Boutilier 1994). Specificity principle has been commonly used to characterize such a preorder. Proposition 1 summarizes existing results about the uniqueness of models (which are complete preorders) for each semantics.

Proposition 1 *(Kaci and van der Torre 2008) (1) The least specific model of $\mathcal{P}_{\rhd_{opt}} \cup \mathcal{P}_{\unrhd_{opt}}$ (resp. $\mathcal{P}_{\rhd_{st}} \cup \mathcal{P}_{\unrhd_{st}}$) exists. (2) The most specific model of $\mathcal{P}_{\rhd_{pes}} \cup \mathcal{P}_{\unrhd_{pes}}$ (resp. $\mathcal{P}_{\rhd_{st}} \cup \mathcal{P}_{\unrhd_{st}}$) exists. (3) The most (resp. least) specific model of $\mathcal{P}_{\rhd_{opt}} \cup \mathcal{P}_{\unrhd_{opt}}$ (resp. $\mathcal{P}_{\rhd_{pes}} \cup \mathcal{P}_{\unrhd_{pes}}$) does not exist. (4) The least/most specific models do not exist for $\mathcal{P}_{\rhd_{opp}} \cup \mathcal{P}_{\unrhd_{opp}}$.*

For non-strict comparative preference statements only both the least and most specific models exist, and they are the trivial preference relation in which all outcomes are equivalent. Thus, the notion of maximal and minimal specificity for preference sets consisting of non-strict comparative preference statements only is not very useful. Besides, the authors of (Kaci and van der Torre 2008) have shown that some semantics can be used together while preserving the uniqueness of the models. More precisely, the least (resp. most) specific model of $\mathcal{P}_{\rhd_{st}} \cup \mathcal{P}_{\unrhd_{st}} \cup \mathcal{P}_{\rhd_{opt}} \cup \mathcal{P}_{\unrhd_{opt}}$ (resp. $\mathcal{P}_{\rhd_{st}} \cup \mathcal{P}_{\unrhd_{st}} \cup \mathcal{P}_{\rhd_{pes}} \cup \mathcal{P}_{\unrhd_{pes}}$) exists.

In this paper we are looking for unique models so we will no longer consider opportunistic semantics. Due to space limitation we do not give algorithms to compute the unique models and refer the reader to (Kaci and van der Torre 2008).

Example & Problem

Example 1 *Suppose an individual is planning a holiday. She/he expresses her/his preferences on the basis of three variables: P (for period) which is either W or S (Winter and Summer resp.), D (for destination) which is either M or B (Mountain and Beach resp.) and L (for location) which is either H or A (Hotel and Apartment resp.). Therefore we have $Dom(P) = \{W, S\}$, $Dom(D) = \{M, B\}$, $Dom(L) = \{H, A\}$ and $\Omega = \{\omega_0 : HMW, \omega_1 : HMS, \omega_2 : HBW, \omega_3 : HBS, \omega_4 : AMW, \omega_5 : AMS, \omega_6 : ABW, \omega_7 : ABS\}$. The individual expresses five preference statements: (i) she/he would prefer travel in summer than in winter, (ii) if destination is mountain then she/he would prefer travel in winter than in summer, (iii) if she/he travels in winter than she/he would prefer rent an apartment than a hotel, (iv) if she/he goes to a hotel then she/he would prefer beach to mountain and (v) if she/he goes to a hotel then summer and winter have equal preference. Formally we write $\mathcal{P} = \{s_1 : S \rhd_* W, s_2 : M \wedge W \rhd_* M \wedge S, s_3 : W \wedge A \rhd_* W \wedge H, s_4 : H \wedge B \rhd_* H \wedge M, s_5 : H \wedge S \unrhd_* H \wedge W, s_6 : H \wedge W \unrhd_* H \wedge S\}$ (s_i stands for "statement"). We have $\mathcal{L}(\mathcal{P}_*) = \{C_1 = (\{\omega_1, \omega_3, \omega_5, \omega_7\}, \{\omega_0, \omega_2, \omega_4, \omega_6\}), C_2 = (\{\omega_0, \omega_4\}, \{\omega_1, \omega_5\}), C_3 = (\{\omega_4, \omega_6\}, \{\omega_0, \omega_2\}), C_4 = (\{\omega_2, \omega_3\}, \{\omega_0, \omega_1\}), C_5 = (\{\omega_1, \omega_3\}, \{\omega_0, \omega_2\}), C_6 = (\{\omega_0, \omega_2\}, \{\omega_1, \omega_3\})\}$.*
*Let *=opt. The associated preference relation is $\succeq = (\{\omega_7\}, \{\omega_4, \omega_6\}, \{\omega_2, \omega_3, \omega_5\}, \{\omega_0, \omega_1\})$. Therefore we have that ω_4 and ω_6 (resp. $\omega_2, \omega_3, \omega_5$ and ω_0, ω_1) are equally preferred. However only ω_2 and ω_3 are equally preferred due to s_5 and s_6. All other equalities are incomparability turned into equality due to specificity principle. This information may be important in recommender systems explaining whether the preferred outcomes (or any two outcomes) are in fact equally preferred or incomparable.*

Qualitative Constraint Satisfaction Problems on Partially Ordered Sets

In this section, we introduce particular qualitative constraint satisfaction problems which will allow to reason about preference statements. For this purpose, let us first introduce the relations of the Point Algebra and the $\mathsf{min}/\mathsf{max}$ relations.

The Point Algebra (Broxvall and Jonsson 2003), PA for short, is a qualitative formalism for representing and reasoning about qualitative temporal constraints (Ligozat and Renz 2004). Given a partial preorder $\geq$ over a set T, denoted by $(T, \geq)$, PA considers four basic relations:

- $\forall x, y \in T, x > y$ iff $x \geq y$ and not$(y \geq x)$;
- $\forall x, y \in T, x < y$ iff $y \geq x$ and not$(x \geq y)$;
- $\forall x, y \in T, x = y$ iff $x \geq y$ and $y \geq x$;
- $\forall x, y \in T, x \| y$ iff neither $x \geq y$ nor $y \geq x$.

The set of these basic relations will be denoted by B_{PA} in the sequel. Note that these relations are jointly exhaustive and pairwise disjoint (*i.e.* any pair of elements x and y belonging to T satisfies exactly one relation of B_{PA}). Making the parallel with a preference relation $\succeq$, the above basic relations are strict preference, equality and incomparability.

A (complex) relation of PA corresponds to a subset of basic relations of B_{PA}. It is represented by this subset. For example, the set $\{\|, =\}$ corresponds to the relation $\| \cup =$. It is satisfied by two elements x and y if and only if $x \| y$ or $x = y$. $2^{\mathsf{B}_{\mathsf{PA}}}$ will denote the set of the 16 relations of PA.

We now need to translate Definition 2 in the setting of constraint satisfaction problems. Given a preorder $(T, \geq)$ and a subset $X \subseteq T$, we define $\mathsf{min}(X)$ and $\mathsf{max}(X)$ in the following way: $\mathsf{min}(X) = \{x \in X : \nexists y \in X$ such that $x \geq y$ and not$(y \geq x)\}$ and $\mathsf{max}(X) = \{x \in X : \nexists y \in X$ such that $y \geq x$ and not$(x \geq y)\}$. We now define 8 binary relations between subsets of T which we call $\mathsf{min}/\mathsf{max}$ basic relations. Each is denoted by rel^{op1}_{op2} with $rel \in \{\geq, >\}$ and $op1, op2 \in \{\mathsf{min}, \mathsf{max}\}$. They are defined as follows:

- for $op1, op2 \in \{\mathsf{min}, \mathsf{max}\}$, $\forall X, Y \in 2^T$, $X >^{op1}_{op2} Y$ iff $\forall x \in op1(X)$ and $\forall y \in op2(Y)$, $x \geq y$ and not$(y \geq x)$;
- for $op1, op2 \in \{\mathsf{min}, \mathsf{max}\}$, $\forall X, Y \in 2^T$, $X \geq^{op1}_{op2} Y$ iff $\forall x \in op1(X)$ and $\forall y \in op2(Y)$, $x \geq y$.

In the sequel we will denote by $\mathsf{B}_{\mathsf{min/max}}$ the set composed of these 8 basic relations, namely $\mathsf{B}_{\mathsf{min/max}} = \{>^{\mathsf{min}}_{\mathsf{min}}, >^{\mathsf{min}}_{\mathsf{max}}, >^{\mathsf{max}}_{\mathsf{min}}, >^{\mathsf{max}}_{\mathsf{max}}, \geq^{\mathsf{min}}_{\mathsf{min}}, \geq^{\mathsf{min}}_{\mathsf{max}}, \geq^{\mathsf{max}}_{\mathsf{min}}, \geq^{\mathsf{max}}_{\mathsf{max}}\}$.

We are now ready to define qualitative constraint satisfaction problems which we call Qualitative Constraint Problems on Partially Ordered Sets ($\mathsf{QCP_{POS}}$ for short). A $\mathsf{QCP_{POS}}$ $\mathcal{Q}$ allows to specify possible configurations of partially ordered elements represented by a set of variables $\mathcal{V}$. It is composed of two types of qualitative constraints. The first type is represented by a map c allowing to specify the relative ordering of each pair of elements of $\mathcal{V}$ by means of a relation of PA. The second type is collected in a set $\mathcal{C}$ and apply to subsets of $\mathcal{V}$. More precisely, each constraint of $\mathcal{C}$ constrains the relative ordering between the minima and the maxima of two subsets of elements. It is of the form $V\ R\ V'$ with $V, V' \in 2^{\mathcal{V}}$ and $R \subseteq \mathsf{B}_{\mathsf{min/max}}$. In the sequel, such constraints will be called min/max constraints over $\mathcal{V}$ and will correspond to a set denoted by $\mathcal{C}^{\mathcal{V}}_{\mathsf{min/max}}$. Formally, a $\mathsf{QCP_{POS}}$ is defined as follows:

Definition 4 *A $\mathsf{QCP_{POS}}$ is a triple $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$ where:*
- *$\mathcal{V}$ is a non-empty finite set of variables;*
- *c is a mapping that associates a relation $c(v, v') \in 2^{\mathsf{B_{PA}}}$ with each pair (v, v') of $V \times V$. c is such that $c(v, v) \subseteq \{=\}$ and $c(v, v') = (c(v', v))^{-1}$ for every $v, v' \in V$ (where $^{-1}$ denotes the inverse operation of PA);*
- *$\mathcal{C}$ is a finite subset of $\mathcal{C}^{\mathcal{V}}_{\mathsf{min/max}}$. We will suppose that for each min/max constraint $V\ R\ V' \in \mathcal{C}$, V and V' are non-empty sets.*

We say that a $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$ is atomic iff for all $v, v' \in \mathcal{V}$, the relation $c(v, v')$ is defined by a singleton relation of PA. Moreover, a $\mathsf{QCP_{POS}}$ $\mathcal{Q}' = (\mathcal{V}', c', \mathcal{C}')$ is a sub-$\mathsf{QCP_{POS}}$ of $\mathcal{Q}$ when $\mathcal{V} = \mathcal{V}'$, for all $v, v' \in \mathcal{V}$ $c'(v, v') \subseteq c(v, v')$, $|\mathcal{C}| = |\mathcal{C}'|$ and for each min/max constraint $X\ R'\ Y \in \mathcal{C}'$ there exists $X\ R\ Y \in \mathcal{C}$ with $R' \subseteq R$.

Definition 5 *Let $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$. An interpretation π of $\mathcal{Q}$ is a pair $(f, (T, \geq))$ where f is a bijection from $\mathcal{V}$ to T and $(T, \geq)$ a partial preorder. An interpretation $\pi = (f, (T, \geq))$ of $\mathcal{Q}$ is a solution of $\mathcal{Q}$ iff :*
- *for each pair of variables $v, v' \in \mathcal{V}$, $f(v)\ c(v, v')\ f(v')$, i.e. the pair $(f(v), f(v'))$ satisfies a basic relation of $\mathsf{B_{PA}}$ belonging to $c(v, v')$;*
- *for each constraint $V\ R\ V' \in \mathcal{C}$, $f(V)$ and $f(V')$ satisfy at least one min/max basic relation of the set R.*

On the other hand, two $\mathsf{QCP_{POS}}$ defined on a same set of variables are equivalent iff they admit the same solutions. Given a $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$, the pair $(\mathcal{V}, c)$ corresponds to a constraint problem of the PA. Hence, a $\mathsf{QCP_{POS}}$ can be seen as a particular generalization of a qualitative problem of the Point Algebra. Moreover, we can note that the constraints of c can be expressed by means of constraints belonging to $\mathcal{C}^{\mathcal{V}}_{\mathsf{min/max}}$ and constraining singleton sets. For example, the constraint $v\ \{>, =\}\ v'$ can be equivalently formulated by the min/max constraint $\{v\} \geq^{\mathsf{min}}_{\mathsf{min}} \{v'\}$. As another example, consider the PA constraint $v\ \{=\}\ v'$. This constraint can be expressed by the conjunction of the two min/max constraints $\{v\} \geq^{\mathsf{min}}_{\mathsf{min}} \{v'\}$ and $\{v'\} \geq^{\mathsf{min}}_{\mathsf{min}} \{v\}$. Therefore, each $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$ can be equivalently expressed by a $\mathsf{QCP_{POS}}$ $\mathcal{Q}' = (\mathcal{V}, c', \mathcal{C}')$ such that $c'(v, v') = \mathsf{B_{PA}}$ for each pair of distinct variables (v, v'). Nevertheless, c allows to express possible ordering relations between every pair of variables of $\mathcal{V}$. Given a $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$, different problems may arise:

- *The consistency problem*: decide whether $\mathcal{Q}$ is consistent or not, *i.e.* does $\mathcal{Q}$ admit at least one solution? Note that given that $\mathsf{QCP_{POS}}$s generalize constraint problems defined on PA, we can establish the NP-completeness of the consistency problem of $\mathsf{QCP_{POS}}$s.
- *The problem of finding a solution of* $\mathcal{Q}$. This problem means characterizing an equivalent sub-$\mathsf{QCP_{POS}}$ $\mathcal{Q}'$ of $\mathcal{Q}$ such that $\mathcal{Q}'$ is consistent and atomic. From such a $\mathsf{QCP_{POS}}$ $\mathcal{Q}'$ we can easily deduce a solution of $\mathcal{Q}$.
- *The minimal labeling problem*: find for every pair of variables $(v, v') \in \mathcal{V} \times \mathcal{V}$ and every constraint of $X\ R\ Y \in \mathcal{C}$ the set of feasible basic relations of $c(v, v')$ and R, *i.e.* the set of basic relations of $c(v, v')$ and R involved in at least a solution. This problem consists in computing the minimal $\mathsf{QCP_{POS}}$ of $\mathcal{Q}$, *i.e.* the unique sub-$\mathsf{QCP_{POS}}$ of $\mathcal{Q}$ equivalent to $\mathcal{Q}$ and having the property of minimality (every basic relation of each of its constraints is feasible).

A method for finding a consistent and atomic sub-$\mathsf{QCP_{POS}}$ of a $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$ in case it exists consists of a backtrack search on the PA part of $\mathcal{Q}$ (*i.e.* the constraint problem of PA defined by $\mathcal{N} = (\mathcal{V}, c)$). At each step of the search, a non-singleton constraint $c(v, v')$ is selected and defined by a relation composed of one of its basic relations. Moreover, path-consistency method is applied to filter the search space and to ensure the consistency of the built partial atomic qualitative constraint network of PA. Note that in the context of qualitative formalisms, path-consistency method is a polynomial method which is usually used to remove unfeasible basic relations from the operation of composition, see (Anger, Mitra, and Rodríguez 1999) for example. During the search, to ensure that the built consistent atomic qualitative constraint network of PA can be extended to a consistent atomic sub-$\mathsf{QCP_{POS}}$ of $\mathcal{Q}$ we must remove unfeasible basic relations of each constraint belonging to $\mathcal{C}$.

From Preferences to Qualitative Constraints

In this section, we are going to show that $\mathsf{QCP_{POS}}$ can be used in a natural way to represent and reason about preference statements. Consider a set of outcomes Ω and a set of preference statements $\mathcal{P} = \{s_0, \ldots, s_k\}$. We will denote by $*(s_i)$, with $i \in \{0, \ldots, k\}$, the semantics associated with the statement s_i ($*(s_i) \in \{st, opt, pes, opp\}$) and by $\mathsf{rel}(s_i)$ the strict/non-strict preference relation used in s_i ($\mathsf{rel}(s_i) \in \{\rhd, \unrhd\}$). We define $\mathsf{QCP_{POS}}(\Omega, \mathcal{P})$ by $\mathsf{QCP_{POS}}$ $(\mathcal{V}, c, \mathcal{C})$ such that:

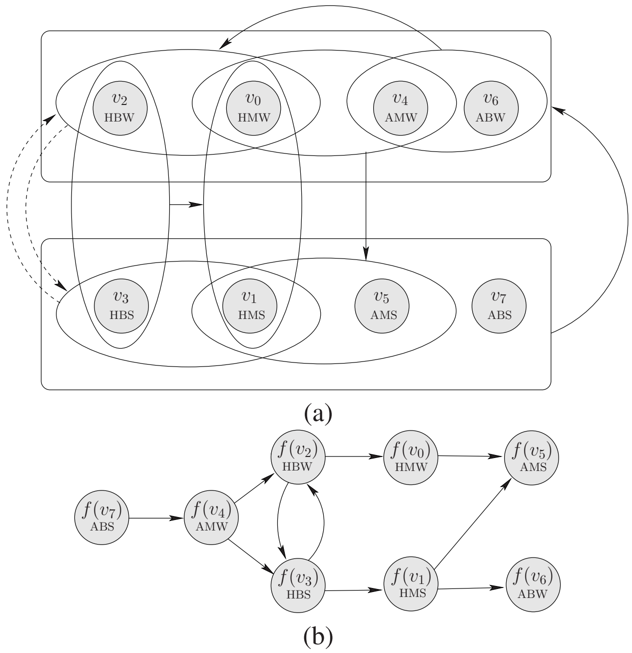

Figure 1: (a) The constraints $\mathcal{C}$ of the $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C}) = \mathsf{QCP_{POS}}(\Omega, \mathcal{P})$ with $\mathcal{P}$ the preference statements over Ω of Example 1, (b) a solution $(f, \geq)$ of $\mathcal{Q}$.

- With each $w_i \in \Omega$ is associated a variable $v_i \in \mathcal{V}$. Hence, for $\Omega = \{w_0, \dots, w_k\}$, $\mathcal{V}$ is defined by the set $\{v_1, \dots, v_k\}$.
- For each variable $v, v' \in \mathcal{V}$ with $v \neq v'$, $c(v, v')$ is defined by the total relation of PA, *i.e.* the relation $\mathsf{B_{PA}}$. For each $v \in \mathcal{V}$, $c(v, v)$ is the relation $\{=\}$.
- For each preference statement $s_i \in \mathcal{P}$ is defined a $\mathsf{min/max}$ constraint $V_i \ R_i \ V'_i \in \mathcal{C}$. V_i and V'_i correspond to the two subsets of variables of $\mathcal{V}$ representing the two subsets of outcomes linked by s_i. R_i is a set composed of a unique basic relation of $\mathsf{B_{min/max}}$. This relation depends on the semantics $*(s_i)$ and the strict/non-strict preference relation $\mathsf{rel}(s_i)$. More precisely, for $*(s_i) = st$ (resp. $*(s_i) = opt$, $*(s_i) = pes$, $*(s_i) = opp$), in case $\mathsf{rel}(s_i)$ is $\rhd$, R_i is defined by $\{>^{\mathsf{min}}_{\mathsf{max}}\}$ (resp. $\{>^{\mathsf{max}}_{\mathsf{max}}\}$, $\{>^{\mathsf{min}}_{\mathsf{min}}\}$, $\{>^{\mathsf{max}}_{\mathsf{min}}\}$). In case $\mathsf{rel}(s_i)$ is $\unrhd$, R_i is defined by $\{\geq^{\mathsf{min}}_{\mathsf{max}}\}$ (resp. $\{\geq^{\mathsf{max}}_{\mathsf{max}}\}$, $\{\geq^{\mathsf{min}}_{\mathsf{min}}\}$, $\{\geq^{\mathsf{max}}_{\mathsf{min}}\}$) .

Example 2 *(Example 1 continued) Consider again Ω and $\mathcal{P}$. Define $\mathsf{QCP_{POS}}$ $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C})$ by $\mathcal{Q} = \mathsf{QCP_{POS}}(\Omega, \mathcal{P})$. In Figure 1.a are represented the* $\mathsf{min/max}$ *constraints of the set $\mathcal{C}$. A solid arrow corresponds to the set $\{>^{\mathsf{max}}_{\mathsf{max}}\}$ whereas a dotted arrow corresponds to $\{\geq^{\mathsf{max}}_{\mathsf{max}}\}$. For example, we have the two* $\mathsf{min/max}$ *constraints $\{v_2, v_3\}\{>^{\mathsf{max}}_{\mathsf{max}}\}\{v_0, v_1\}$ and $\{v_0, v_2\}\{\geq^{\mathsf{max}}_{\mathsf{max}}\}\{v_1, v_3\}$ belonging to the set $\mathcal{C}$.*

Now, we are going to study how to reason about a set of preference statements from its representation into a $\mathsf{QCP_{POS}}$. First of all, we can clearly establish that the consistency (or non consistency) of a set of preference statements can be decided from the consistency of its corresponding $\mathsf{QCP_{POS}}$. Indeed, we have the following property:

Proposition 2 *Let Ω be a set of outcomes and $\mathcal{P}$ be a set of preference statements. $\mathcal{P}$ is consistent if and only if $\mathsf{QCP_{POS}}(\Omega, \mathcal{P})$ is consistent.*

Proofs are omitted due to space limitation.
For illustration, a solution $(f, (T, \geq))$ of $\mathsf{QCP_{POS}}(\Omega, \mathcal{P})$, with $\mathcal{P}$ the set of preference statements given in Example 1, is represented by an oriented graph in Figure 1.b. An oriented edge $(f(v), f(v'))$ means $f(v) \geq f(v')$. $f(v) \geq f(v')$ is not represented in case where v and v' are identical variables or when it can be deduced by transitivity from oriented edges already represented. From the previous proposition, we can assert that $\mathcal{P}$ is a consistent set of preference statements. Note also that from the solution $(f, (T, \geq))$ we can easily define a model of $\mathcal{P}$.

As previously noticed, for some preference semantics we can characterize unique models following specificity principles. Now, we are going to show how these models can be characterized from resolutions of $\mathsf{QCP_{POS}}$s. More specifically, we provide an algorithm allowing to compute the least specific model (resp. the most specific model) of a set of preference statements given strong semantics or optimistic semantics (resp. pessimistic semantics) in case it exists, i.e. the set of preference statements is consistent.

Algorithm 1: specificModel(Ω,$\mathcal{P}$,kind)

In : A set of outcomes Ω, a set $\mathcal{P}$ of preference statements, the kind of the unique model with $\mathsf{kind} \in \{\mathsf{leastSpecific}, \mathsf{mostSpecific}\}$.
Out : A unique model represented by a sequence $(E_0, \dots, E_n)$ or $\mathsf{Inconsistent}$.

begin
  if $\mathsf{kind} = \mathsf{leastSpecific}$ **then**
    $rel \leftarrow \{>, =\}$;
  else
    $rel \leftarrow \{<, =\}$;
  $\mathcal{Q} = (\mathcal{V}, c, \mathcal{C}) \leftarrow \mathsf{QCP_{POS}}(\Omega, \mathcal{P})$; $k \leftarrow 0$; $\mathcal{V}' \leftarrow \emptyset$;
  while $\mathcal{V} \neq \mathcal{V}'$ **do**
    $E_k \leftarrow \emptyset$;
    for $v_i \in \mathcal{V} \setminus \mathcal{V}'$ **do**
      for $v \in \mathcal{V} \setminus (\mathcal{V}' \cup \{v_i\})$ **do**
        $c(v_i, v) \leftarrow rel$; $c(v, v_i) \leftarrow rel^{-1}$;
      if $\mathsf{consistent}(\mathcal{Q})$ **then**
        $E_k \leftarrow E_k \cup \{w_i\}$; $\mathcal{V}' \leftarrow \mathcal{V}' \cup \{v_i\}$;
      else
        for $v \in \mathcal{V} \setminus (\mathcal{V}' \cup \{v_i\})$ **do**
          $c(v_i, v) \leftarrow \mathsf{B_{PA}}$; $c(v, v_i) \leftarrow \mathsf{B_{PA}}$;
    if $E_k = \emptyset$ **then**
      return $\mathsf{Inconsistent}$**;**
    $k \leftarrow k + 1$;
  if $\mathsf{kind} = \mathsf{leastSpecific}$ **then**
    return $(E_0, \dots, E_{k-1})$ **;**
  else
    return $(E_{k-1}, \dots, E_0)$ **;**

Consider the function specificModel which takes as parameters a set of outcomes Ω, a set of preference statements $\mathcal{P}$ and the kind of the unique model we want. Roughly speaking, by considering $\mathcal{Q} = \mathsf{QCP}_{\mathsf{POS}}(\Omega, \mathcal{P})$ this function builds the partition of Ω corresponding to this model. For the least (resp. most) specific model, the strata of lower (resp. greater) ranks are considered first. For a given outcome ω_i, its membership to a given stratum is decided by testing the consistency of a sub-$\mathsf{QCP}_{\mathsf{POS}}$ of $\mathcal{P}$ constraining the position of ω_i in the stratum. This is done by adding precedence constraints on v_i defined by $\{<, =\}$ (resp. $\{>, =\}$). We can formally prove the following property:

Proposition 3 *Let Ω be a set of outcomes. Let $\mathcal{P}$ be a set of preference statements interpreted following a given semantics $*$. The call of the function* specificModel *with $\Omega, \mathcal{P}$ and* leastSpecific *(resp.* mostSpecific*) as parameters returns for the semantics $* \in \{st, opt\}$ (resp. $* \in \{st, pes\}$) the least specific model (resp. most specific model) in case it exists.*

Usually, for decision purposes it is more convenient to consider a unique model. Nevertheless, due to the fact such a model is a complete preorder on Ω we cannot distinguish between situations of equality and situations of incomparability over two outcomes belonging to the same partition (*i.e.* two outcomes $\omega, \omega' \in \Omega$ equal w.r.t. the least/most specific model). A way to remedy this shortcoming is to see the least/most specific model as a set of particular models and to consider it as complementary constraints. This leads us to consider a new translation of a set of preference statements into $\mathsf{QCP}_{\mathsf{POS}}$ which takes into account the least/most specific model. We will denote it by $\mathsf{QCP}'_{\mathsf{POS}}$. Let $\mathcal{P}$ a set of preference statements and a sequence $\succeq = (E_0, \ldots, E_n)$ corresponding to the least/most specific model of $\mathcal{P}$. $\mathsf{QCP}'_{\mathsf{POS}}(\Omega, \mathcal{P}, \succeq)$ is defined by the $\mathsf{QCP}_{\mathsf{POS}}$ $(\mathcal{V}, c, \mathcal{C})$ where $\mathcal{V}$ and $\mathcal{C}$ are respectively the set of variables and the set of $\mathsf{min}/\mathsf{max}$ constraints of $\mathsf{QCP}_{\mathsf{POS}}(\Omega, \mathcal{P})$ and c is defined for each $v_i, v_j \in \mathcal{V}$ by:

- in case $i = j$, $c(v_i, v_i)$ is the relation $\{=\}$;
- in case $i \neq j$ and $v_i, v_j \in E_k$ with $k \in \{0, \ldots, n\}$, $c(v_i, v_j)$ is the relation $\{=, \|\}$;
- in case $i \neq j$, $v_i \in E_k$, $v_j \in E_l$ with $k, l \in \{0, \ldots, n\}$ and $k < l$ (resp. $l > k$), $c(v_i, v_j)$ is $\{>\}$ (resp. $\{<\}$).

Considering $\mathcal{Q} = \mathsf{QCP}'_{\mathsf{POS}}(\Omega, \mathcal{P}, \succeq)$ allows us for example to determine outcomes put together in a same stratum in $\succeq$ because they are equally preferred due to an equal preference statement.

Example 3 *Let $\mathcal{Q}$ the $\mathsf{QCP}_{\mathsf{POS}}$ defined by $\mathcal{Q} = \mathsf{QCP}'_{\mathsf{POS}}(\Omega, \mathcal{P}, \succeq)$ with Ω, $\mathcal{P}$ and $\succeq$ respectively corresponding to the set of outcomes, the set of preference statements and the least specific model given in Example 1. The resolution of the minimal labeling problem of $\mathcal{Q}$ provides the minimal $\mathsf{QCP}_{\mathsf{POS}}$ $\mathcal{Q}_{min} = (\mathcal{V}, c, \mathcal{C})$ with the map c represented by the following table:*

$c(v, v')$	v_7	v_4	v_6	v_2	v_3	v_5	v_0	v_1
v_7	$\{=\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$
v_4	$\{<\}$	$\{=\}$	$\{=, \Vert\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$
v_6	$\{<\}$	$\{=, \Vert\}$	$\{=\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$	$\{>\}$
v_2	$\{<\}$	$\{<\}$	$\{<\}$	$\{=\}$	$\{=\}$	$\{=, \Vert\}$	$\{>\}$	$\{>\}$
v_3	$\{<\}$	$\{<\}$	$\{<\}$	$\{=\}$	$\{=\}$	$\{=, \Vert\}$	$\{>\}$	$\{>\}$
v_5	$\{<\}$	$\{<\}$	$\{<\}$	$\{=, \Vert\}$	$\{=, \Vert\}$	$\{=\}$	$\{>\}$	$\{>\}$
v_0	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{=\}$	$\{=, \Vert\}$
v_1	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{<\}$	$\{=, \Vert\}$	$\{=\}$

The relation $c(v, v')$ is given by the entry whose row corresponds to v and column corresponds to v'. We can for example note that whatever the models satisfying $\mathcal{P}$ the outcomes ω_2 and ω_3 cannot be incomparable ($c(v_2, v_3) = \{=\}$).

Conclusion and Perspectives

Preferences are expressed in conditional logics with strict or non-strict comparative preference statements. They can be interpreted following different semantics. Specificity principles are used to characterize a unique model given a set of preference statements and a semantics. The main limitation of these logics is preference queries. More precisely, two outcomes equally preferred w.r.t. this unique model are not distinguishable. We do not know whether they are equally preferred due to non-strict preference statements or incomparable. In this paper, we solved this problem by encoding preference logics and their associated algorithms in the framework of constraint satisfaction problems.

An analysis of our encoding tells us that $\mathsf{min}/\mathsf{max}$ constraints resulting from the translation of a set of preference statements into a $\mathsf{QCP}_{\mathsf{POS}}$ are defined on the basis of singleton sets of $\mathsf{B}_{\mathsf{min}/\mathsf{max}}$. This is due to the fact that preference statements use a unique comparison operator ($\rhd$ or $\unrhd$). Given the expressiveness power of $\mathsf{QCP}_{\mathsf{POS}}$ we are now able to extend conditional logics and consider preference statements expressing for example a choice between $\rhd$ or $\unrhd$ (i.e., $\alpha \rhd \beta$ or $\beta \rhd \alpha$). This type of preference can be then expressed by means of non-singleton relations of $\mathsf{B}_{\mathsf{min}/\mathsf{max}}$.

References

Anger, F. D.; Mitra, D.; and Rodríguez, R. V. 1999. Satisfiability in nonlinear time: Algorithms and complexity. In *FLAIRS'99*, 406–411.

Benferhat, S.; Dubois, D.; Kaci, S.; and Prade, H. 2002. Bipolar representation and fusion of preferences in the possibilistic logic framework. In *KR'02*, 421–432.

Boutilier, C. 1994. Toward a logic for qualitative decision theory. In *KR'94*, 75–86.

Broxvall, M., and Jonsson, P. 2003. Point algebras for temporal reasoning: Algorithms and complexity. *Artificial Intelligence.* 149(2):179–220.

Kaci, S., and van der Torre, L. 2008. Reasoning with various kinds of preferences: Logic, non-monotonicity and algorithms. *Annals of Operations Research* 163(1):89–114.

Ligozat, G., and Renz, J. 2004. What is a qualitative calculus? a general framework. In *PRICAI'04*.

van der Torre, L., and Weydert, E. 2001. Parameters for utilitarian desires in a qualitative decision theory.*Applied Int.* 14(3):285–301.

von Wright, G. H. 1963. *The Logic of Preference*. University of Edinburgh Press.

Yager, R. 1983. Entropy and specificity in a mathematical theory of evidence. *Int. J. of General Systems* 9:249–260.