Axiomatic Characterization of Game-Theoretic Network Centralities

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Abstract

One of the fundamental research challenges in network science is the centrality analysis, i.e., identifying the nodes that play the most important roles in the network. In this paper, we focus on the game-theoretic approach to centrality analysis. While various centrality indices have been proposed based on this approach, it is still unknown what distinguishes this family of indices from the more classical ones. In this paper, we answer this question by providing the first axiomatic characterization of game-theoretic centralities. Specifically, we show that every centrality can be obtained following the game-theoretic approach, and show that two natural classes of game-theoretic centrality can be characterized by two intuitive properties pertaining to Myerson's notion of Fairness.

Introduction

Centrality analysis is one of the fundamental research problems in graph theory and network analysis. It involves identifying the nodes that play the most important role in the network (Brandes and Erlebach 2005). On top of the already classic centrality indices such as degree, closeness, betweenness, eigenvector, Katz, and PageRank centralities, various new concepts have been recently proposed in the literature.

One family of centralities that has recently attracted growing attention is based on cooperative game theory (Gomez et al. 2003, del Pozo et al. 2011, Michalak et al. 2013). The key idea behind this approach is to analyse the topology of the network using the combinatorial structure of a coalitional game. More in detail, in the first step, one has to define a function that evaluates the centrality of each subset of nodes. Next, having evaluated all the subsets, one can use payoff division schemes from cooperative game theory to measure how individual nodes contribute to subsets' performance. This extends conventional centrality indices which solely focus on the performance of individual nodes. Various centralities obtained using the game-theoretic approach have been shown to perform relatively better than classic approaches in a number of real-life applications, including terrorist-network analysis (Lindelauf, Hamers, and Husslage 2013; Michalak et al. 2015), genes and brain networks (Kötter et al. 2007; Moretti et al. 2010), and energy savings in IT networks (Bianzino et al. 2012).

However, despite the plethora of game-theoretic centralities in the literature, their theoretical foundations and properties are not yet entirely understood. Unfortunately, this problem, termed "*theory gap*" concerns not only novel centrality indices but also the classic ones as well as many other concepts in social network analysis (Schoch and Brandes 2015). A few attempts to bridge this gap include the works by (Sabidussi 1966, Koschützki 2005, Boldi and Vigna 2014) who provided axiomatic characterizations of some of the classic centrality indices.

Meager axiomatic foundations are especially striking in the case of game-theoretic indices. This is because the axiomatizations of the payoff division schemes – the schemes based on which the game-theoretic indices are built – have been extensively studied. Hence, at first glance, it should be straightforward to translate those game-theoretic axiom systems to the network context. This, however, is not the case. It turns our that most axioms that seem desirable in the coalitional game context lose their attractiveness when applied to networks. A notable exception is the *Fairness* axiom introduced by Myerson (1977) for graph-restricted coalitional games; it says that *each edge equally affects the payoff of both adjacent nodes*. Indeed, Fairness is one of the axioms used by Skibski et al. 2016, who proposed the first characterization of a particular game-theoretic centrality.

Nevertheless, little is known about the axiomatic underpinnings of game-theoretic indices in general. In particular, we still do not know how general this approach is, and what distinguishes game-theoretic centrality indices from others.

In what follows, we present the first attempt to answer these question. We show that every non game-theoretic centrality can be also obtained using some game-theoretic centrality; this is a testimony to the versatility of this approach to centrality analysis. Next, we define two natural classes of game-theoretic centrality indices: *separable* and *induced*. We prove that the class of centralities obtained using separable game-theoretic centralities is defined by Myerson's Fairness. In other words, any separable game-theoretic centrality satisfies Fairness, and any (non-game-theoretic) centrality that satisfies Fairness can be obtained with some separable game-theoretic centrality. Next, we extend Fairness to a new axiom, *Edge Balanced Contribution*. Analogously, we prove that the class obtained using separable game-theoretic centralities is defined by Edge Balanced Contribution.

Preliminaries

In this section, we provide the necessary background and notation from both graph theory and coalitional game theory.

Graph theory

A graph (network) is a pair $G = (V, E)$, where V is the set of nodes and E is the set of edges. The edge between any two nodes, $v, u \in V$, will be denoted by $\{u, v\}$. Given a set of nodes, V, the set of all possible graphs will be denoted by $\mathcal{G}^V$. Furthermore, the set of all possible edges will be denoted by $\mathcal{E}^V$, i.e., $\mathcal{E}^V = \{S \subseteq V : |S| = 2\}$.

For any subset of nodes, $S \subseteq V$, the *subgraph of G induced by S* is denoted by $G[S]$ and is defined as the graph whose set of nodes is S and whose set of edges consists of every edge in E of which both ends belong to S. Formally:

$$G[S] = \big(S, \big\{\{v, u\} \in E : v, u \in S\big\}\big)\,.$$

A subgraph is said to be *connected* if there exists a path between every pair of nodes in that subgraph. Furthermore, any such connected subgraph, $G[S]$, is said to be *maximal* if $G[S']$ is disconnected for all $S \subset S'$. We will refer to each maximal connected subgraph as a *component* of G. Also, we will denote by $\mathcal{K}(G)$ the *partition* of V in which every subset induces a component of G.

A *centrality index* is a function, $c : \mathcal{G}^V \to \mathbb{R}^V$, that assigns to every node $v \in V$ a real number reflecting the importance of v in G; this number is called the *centrality* of v. Typically, the higher the centrality, the more important or *central* the node. Given a set of nodes V, the set of all possible centrality indices is denoted by $\mathcal{C}^V$. For every $c \in \mathcal{C}^V$ and every $k \in \mathbb{R}$, we define the centrality index $(k \cdot c)$ as follows: $(k \cdot c)_v(G) = k \cdot c_v(G)$ for all $G \in \mathcal{G}^V$ and $v \in V$. Similarly, for every $c, c' \in \mathcal{C}^V$, we define the centrality index $(c + c')$ as follows: $(c + c')_v(G) = c_v(G) + c'_v(G)$ for all $G \in \mathcal{G}^V$ and $v \in V$.

Coalitional game theory

A *game* is a pair, (N, f), where N is the set of *players* and $f : 2^N \to \mathbb{R}$ is the *characteristic function*, which assigns to each subset of players a real number reflecting its importance. Any subset of players, $S \subseteq N$, is called a *coalition*, and $f(S)$ is called *the value of coalition S*. Typically, $f(\emptyset) = 0$. Given a set of players, N, the set of all possible games is denoted by $\mathcal{F}^N$.

A *solution concept*, $\varphi : \mathcal{F}^N \to \mathbb{R}^N$, is a function that assigns a *payoff* to each player, v, in any given game (N, f); this payoff is denoted by $\varphi_v(f)$. Given a set of players N, the set of all possible solution concepts is denoted by Φ^N.

A fundamental class of solution concepts is *Semivalues* (Dubey, Neyman, and Weber 1981). Let $\beta : \{0, \ldots, |N| - 1\} \to [0, 1]$ be a function such that $\sum_{k=0}^{|N|-1} \beta(k) = 1$. Every such β defines a unique semivalue, φ^β, based on which the payoff of a player $v \in N$ is computed as follows:

$$\varphi^\beta_v(f) = \sum_{S \subseteq N \setminus \{v\}} \frac{\beta(|S|)}{\binom{|N|-1}{|S|}} (f(S \cup \{v\}) - f(S)). \quad (1)$$

Here, the expression $f(S \cup \{v\}) - f(S)$ is known as the *marginal contribution* of player v to coalition S. We will write $\beta^*(k)$ as a shorthand notation for $\beta(k)/\binom{|N|-1}{k}$. A semivalue is said to be *positive* if $\beta(k) > 0$ for every $k \in \{0, \ldots, |N| - 1\}$. Given N, the set of all semivalues is denoted by $\mathcal{SV}^N$, and the set of all *positive* semivalues is denoted by $\mathcal{SV}^N_+$. Thus, $\mathcal{SV}^N_+ \subseteq \mathcal{SV}^N \subseteq \Phi^N$.

Two well-known solution concepts, namely the Shapley value (Shapley 1953) and the Banzhaf index (Banzhaf III 1965), are in fact positive semivalues, with $\beta^{Shapley}(k) = 1/|N|$, and $\beta^{Banzhaf}(k) = \binom{|N|-1}{k}/2^{|N|-1}$.

Game-theoretic centrality indices

We begin with the definition of a *representation function*, $r : \mathcal{G}^V \to \mathcal{F}^V$, which maps every graph whose set of nodes is V onto a cooperative game whose set of players is V. For a graph $G = (V, E)$, the characteristic function of game $r(G)$ is denoted by f^r_G. That is to say, $r(G) = (V, f^r_G)$. Given a set of nodes V, the set of all possible representation functions will be denoted by $\mathcal{R}^V$.

A *Game-Theoretic Centrality Index (GTC)* is a pair, (r, φ), where r is a representation function, and φ is a solution concept. We say that a game-theoretic centrality index, (r, φ), *generates* a centrality index, $[(r, \varphi)] \in \mathcal{C}^V$, computed for every $G \in \mathcal{G}^V$ and every $v \in V$ as follows:

$$[(r, \varphi)]_v(G) = \varphi_v(f^r_G). \quad (2)$$

In words, the centrality $[(r, \varphi)]$ of node v in the graph G equals the payoff of player v in the game $r(G)$ according to the solution concept φ. We say that (r, φ) is *based* on φ. Given a set of nodes, V, the set of all game theoretic centrality indices will be denoted by $\mathcal{GTC}^V$. Formally:

$$\mathcal{GTC}^V = \{(r, \varphi) : r : \mathcal{G}^V \to \mathcal{F}^V, \varphi \in \Phi^V\}.$$

We will refer to $\mathcal{GTC}^V$ as *the general class of game-theoretic centrality indices*. For any given class, $\mathcal{I} \subseteq \mathcal{GTC}^V$, we will write $[\mathcal{I}]$ to denote the set of centrality indices generated by every $(r, \varphi) \in \mathcal{I}$. For instance, we have:

$$[\mathcal{GTC}^V] = \{[(r, \varphi)] : (r, \varphi) \in \mathcal{GTC}^V\}.$$

Furthermore, for any given class, $\mathcal{I} \subseteq \mathcal{GTC}^V$, we will write $\mathcal{I}_+$ to denote the subclass of $\mathcal{I}$ for which the solution concept happens to be a positive semivalue. Likewise, for any $\varphi \in \Phi^V$, we will write $\mathcal{I}_\varphi$ to denote the subclass of $\mathcal{I}$ for which the solution concept happens to be φ. For instance, we have:

$$\begin{aligned}\mathcal{GTC}^V_+ &= \{(r, \varphi) : r : \mathcal{G}^V \to \mathcal{F}^V, \varphi \in \mathcal{SV}^V_+\},\\ \mathcal{GTC}^V_\varphi &= \{(r, \varphi) : r : \mathcal{G}^V \to \mathcal{F}^V\}.\end{aligned}$$

In this paper, we focus on $\mathcal{GTC}^V_+$. Furthermore, we restrict our attention to a fixed set of nodes, V, and so will often omit V from notation such as $\mathcal{E}^V$, $\mathcal{C}^V$, Φ^V, $\mathcal{SV}^V$, and $\mathcal{GTC}^V$.

General Class of Game-Theoretic Centralities

At first glance, it may seem that $[\mathcal{GTC}] \subsetneq \mathcal{C}$. However, as we will establish in Theorem 1, for every centrality index, $c \in \mathcal{C}$, there exists a game-theoretic centrality index, (r, φ), such that $[(r, \varphi)] = c$. The theorem builds upon a *dummy game* – a standard concept in cooperative game theory.

Theorem 1. *For every positive semivalue* $\varphi \in \mathcal{SV}_+$,

$$[\mathcal{GTC}_\varphi] = [\mathcal{GTC}_+] = [\mathcal{GTC}] = \mathcal{C}.$$

Proof. Since $\varphi \in \mathcal{SV}_+$, then $\mathcal{GTC}_\varphi \subseteq \mathcal{GTC}_+$. Based on this, as well as the fact that $[\mathcal{GTC}] \subseteq \mathcal{C}$, we have:

$$[\mathcal{GTC}_\varphi] \subseteq [\mathcal{GTC}_+] \subseteq [\mathcal{GTC}] \subseteq \mathcal{C}.$$

It remains to prove that $\mathcal{C} \subseteq [\mathcal{GTC}_\varphi]$. To put it differently, for every $c \in \mathcal{C}$, it remains to prove that there exists some $r \in \mathcal{R}$ such that $[(r, \varphi)] = c$. To this end, let $c \in \mathcal{C}$ be an arbitrary centrality index, and let us define a representation function $r(G) = (V, f^r_G)$ for every $G \in \mathcal{G}^V$ such that:

$$\forall S \subseteq V, \quad f^r_G(S) = \sum_{v \in S} c_v(G).$$

This is a *dummy game* – a game in which the value of every coalition is the sum of the values of its members. In our case, we set the value of every $v \in V$ to be equal to the centrality of v in graph G according to c. More precisely, we set $f^r_G(\{v\}) = c_v(G)$. Now since the marginal contribution of v to every coalition is equal to $c_v(G)$, then the definition of semivalues (Equation 1) implies that:

$$\forall v \in V, \quad [(r, \varphi)]_v(G) = \varphi_v(f^r_G) = c_v(G).$$

This concludes the proof of Theorem 1. □

Let us illustrate the construction used in the proof of Theorem 1 through the following example.

Example 1. *Consider the* degree centrality, c^D*, defined as:*

$$c^D_v(G) = |\{\{v, u\} \in E : u \in V\}|.$$

Now, given the two possible graphs in $\mathcal{G}^{\{v,u\}}$*, i.e., given* $G_1 = (\{v, u\}, \emptyset)$ *and* $G_2 = (\{v, u\}, \{\{v, u\}\})$*, let us show how to generate* c^D *using some GTC. First, let us deal with* G_1*. We need to specify* $f^r_{G_1}$ *such that* $\varphi_v(f^r_{G_1}) = c^D_v(G_1) = 0$*, and* $\varphi_u(f^r_{G_1}) = c^D_u(G_1) = 0$*. To this end, let us define a dummy game in which* $f^r_{G_1}(\{v\}) = c^D_v(G_1) = 0$ *and* $f^r_{G_1}(\{u\}) = c^D_u(G_1) = 0$*. This implies that* $f^r_{G_1}(\{v, u\}) = f^r_{G_1}(\{v\}) + f^r_{G_1}(\{u\}) = 0$*. Since every marginal contribution of* v *equals* 0 *(i.e.,* $f^r_{G_1}(\{v\}) - f^r_{G_1}(\emptyset) = 0$ *and* $f^r_{G_1}(\{v, u\}) - f^r_{G_1}(\{u\}) = 0$*), then from the definition of semivalues we get:* $\varphi_v(f^r_{G_1}) = 0$*. Following the same reasoning, we get:* $\varphi_u(f^r_{G_1}) = 0$*. Moving on to* G_2*, we define a dummy game in which* $f^r_{G_2}(\{v\}) = f^r_{G_2}(\{u\}) = 1$*, which implies that* $f^r_{G_2}(\{v, u\}) = 1 + 1 = 2$*. Following the above reasoning, we get:* $\varphi_v(f^r_{G_2}) = \varphi_u(f^r_{G_2}) = 1$*.*

Next, we lay the theoretical foundation for the coming sections by showing that the totality of all centrality indices form a vector space. To this end, let us introduce the class of *unanimity* centrality indices.

Definition 1. *(Unanimity Centrality Indices) Given a set of edges,* $E^\dagger \subseteq \mathcal{E}^V$*, and a set of nodes,* $U \subseteq V$*, the* unanimity centrality index $c^{\langle U, E^\dagger \rangle}$ *is defined for every* $G = (V, E) \in \mathcal{G}^V$ *and every* $v \in V$ *as follows:*

$$c^{\langle U, E^\dagger \rangle}_v(G) = \begin{cases} 1 & \text{if } v \in U \text{ and } E^\dagger \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

As such, $c^{\langle U, E^\dagger \rangle}$ *assigns a value of* 1 *if and only if the node belongs to* U *and the graph contains every edge from* $E^\dagger$*.*

The set of all unanimity centrality indices will be denoted by $\mathcal{U}^V$, or simply $\mathcal{U}$ when there is no risk of confusion. The next lemma provides a sufficient condition for the linear independence of the class of unanimity centrality indices.

Lemma 2. *Let* $\mathcal{U}^*$ *be a set of unanimity centrality indices such that for every set of edges,* $E^\dagger \subseteq \mathcal{E}^V$*, and every pair,* $c^{\langle U, E^\dagger \rangle}, c^{\langle U', E^\dagger \rangle} \in \mathcal{U}^*$*, we have:* $U = U'$ *or* $U \cap U' = \emptyset$*. Then,* $\mathcal{U}^*$ *is linearly independent.*

Next, we use Lemma 2 to characterize a basis of the class of all centrality indices, $\mathcal{C}$.

Theorem 3. *The class* $\mathcal{C}$ *is a vector space with the basis:*

$$\mathcal{U}^V_{All} = \{c^{\langle \{v\}, E^\dagger \rangle} : v \in V, E^\dagger \subseteq \mathcal{E}^V\}.$$

Proof. Since $\mathcal{C}$ is closed under addition (for every $c, c' \in \mathcal{C}$ we have $c + c' \in \mathcal{C}$) and closed under scalar multiplication (for every $c \in \mathcal{C}$ and a scalar $k \in \mathbb{R}$ we have $k \cdot c \in \mathcal{C}$), then $\mathcal{C}$ is a vector space. It remains to prove that $\mathcal{U}_{All}$ is a basis of $\mathcal{C}$. We know from Lemma 2 that $\mathcal{U}_{All}$ is linearly independent. Moreover, since $|\mathcal{U}_{All}| = |V \times \mathcal{E}| = |V \times \mathcal{G}|$, the size of $\mathcal{U}$ is the same as the dimension of $\mathcal{C}$. This concludes the proof. □

The above result comes in handy when proving that all centrality indices from a given class can be generated with a subclass of GTCs. More in detail, the following lemma shows that if the basis of a class can be generated, then the whole class can also be generated.

Lemma 4. *Let* $\mathcal{C}^*$ *be a class of centrality indices with a basis* $\mathcal{U}^*$*, and let* $\mathcal{I} \subseteq \mathcal{GTC}_\varphi$ *be a class of GTCs closed under addition and scalar multiplication. If* $\mathcal{U}^* \subseteq [\mathcal{I}]$*, then* $\mathcal{C}^* \subseteq [\mathcal{I}]$*.*

In the general class of game-theoretic centrality indices, for any two distinct graphs, $G, G' \in \mathcal{G}$, a representation function, r, may output two games, $(V, f^r_G), (V, f^r_{G'}) \in \mathcal{F}$, that are completely independently from one another. In other words, the value of every subset of nodes $S \subseteq V$ under f^r_G may be completely different than (or independent from) the value of the same subset under $f^r_{G'}$. This implies that in order to define a game-theoretic centrality in the general form, one needs to specify all 2^V values of f^r_G for every $G \in \mathcal{G}$. Such centrality index would clearly be impractical. To overcome this limitation, every game-theoretic centrality index studied in the literature to date assumes some kind of dependency between f^r_G and $f^r_{G'}$ (see, e.g., Michalak et al. 2013). We follow this approach in the next sections, where we define two classes of game-theoretic centralities by imposing some natural requirements on the representation function.

Separable Game-Theoretic Centralities

The first subclass of $\mathcal{GTC}$ that we consider is the class of *separable* game-theoretic centralities.

Definition 2. *(Separable GTC ($\mathcal{SGTC}$)) A representation function,* r*, is* separable *if for every coalition* $S \subseteq V$ *and every two graphs* $G, G' \in \mathcal{G}^V$ *such that* $G[S] = G'[S]$ *and*

$G[V \setminus S] = G'[V \setminus S]$ *it holds that* $f^r_G(S) = f^r_{G'}(S)$*. A GTC,* (r, φ)*, is separable if* r *is separable. Given a set of nodes,* V*, the set of all separable GTCs is denoted by* $\mathcal{SGTC}^V$*.*

In words, a game-theoretic centrality index is separable if the value of every coalition, $S \subseteq V$, under the representation function, r, depends solely on the subgraph induced by S and the subgraph induced by $V \setminus S$ in G. As we will show later on in this section, separable GTCs are related to the notion of *Fairness*, proposed by Myerson (1977).

> ***Fairness:*** *For every* $G = (V, E)$ *and every* $v, u \in V$ *such that* $\{v, u\} \notin E$*, adding the edge* $e = \{v, u\}$ *to the graph* G *affects the centrality of* v *and* u *equally. Formally:*
>
> $$c_v((V, E \cup \{e\})) - c_v(G) = c_u((V, E \cup \{e\})) - c_u(G).$$

The class of all centralities satisfying Fairness will be denoted by $\mathcal{C}^V_{Fair}$, or simply $\mathcal{C}_{Fair}$ when there is no risk of confusion.

For every node, $v \in V$, and edge, $e \in \mathcal{E}$, we will use the notation: $\Delta^c_v(e, G) = c_v((V, E \cup \{e\})) - c_v(G)$. Note that, if $e \in E$, then $\Delta^c_v(e, G) = 0$.

The following lemma states that any centrality index in $\mathcal{C}_{Fair}$ can be uniquely characterized by only specifying the sum of node-centralities in every component of the graph (i.e., there is no need to specify the centrality of every node).

Lemma 5. *For every function,* $g : 2^V \times \mathcal{G}^V \to \mathbb{R}$*, there exists at most one centrality index,* $c \in \mathcal{C}^V_{Fair}$*, that satisfies* $\sum_{v \in S} c_v(G) = g(S, G)$ *for every* $G \in \mathcal{G}^V$ *and* $S \in \mathcal{K}(G)$*.*

Building upon Lemma 5, the following theorem identifies a basis of the class $\mathcal{C}_{Fair}$.

Theorem 6. $\mathcal{C}_{Fair}$ *is a vector space with the basis:*

$$\mathcal{U}^V_{Fair} = \{c^{\langle U, E^\dagger \rangle} \in \mathcal{U}^V : U \in \mathcal{K}((V, E^\dagger))\}. \quad (3)$$

Sketch of Proof. We begin by showing that $\mathcal{C}_{Fair}$ is a vector space. To this end, note that if the addition of an edge $\{v, u\}$ affects the centrality of v and u equally according to c and according to c', then it also affects the centrality of v and u equally according to $c + c'$. As such, $\mathcal{C}_{Fair}$ is closed under addition. Analogously, multiplying c by a scalar $k \in \mathbb{R}$ does not violate Fairness, meaning that $\mathcal{C}_{Fair}$ is closed under scalar multiplication. Thus, $\mathcal{C}_{Fair}$ is a vector space.

It remains to prove that $\mathcal{U}_{Fair}$ forms a basis of $\mathcal{C}_{Fair}$. To this end, we will show that: (1) $\mathcal{U}_{Fair} \subseteq \mathcal{C}_{Fair}$, (2) $\mathcal{U}_{Fair}$ is linearly independent, and (3) $|\mathcal{U}_{Fair}|$ is equal to the dimension of $\mathcal{C}_{Fair}$.

As for (1), let $c^{\langle U, E^\dagger \rangle}$ be an arbitrary centrality index in $\mathcal{U}_{Fair}$. The edge $\{v, u\}$ affects the centrality of v in $c^{\langle U, E^\dagger \rangle}$ only if $v \in U$ and $\{v, u\} \in E^\dagger$. This implies that $u \in U$ and that $\Delta^c_v(\{v, u\}, G) = \Delta^c_u(\{v, u\}, G)$. We have shown that an arbitrary $c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{Fair}$ satisfies Fairness, meaning that $\mathcal{U}_{Fair} \subseteq \mathcal{C}_{Fair}$.

As for (2), the linear independence of $\mathcal{U}_{Fair}$ is implied by Lemma 2.

As for (3), let $dim(\mathcal{C}_{Fair})$ denote the dimension of $\mathcal{C}_{Fair}$. Linear independence implies that $dim(\mathcal{C}_{Fair}) \geq |\mathcal{U}_{Fair}|$. But from Lemma 5 and the fact that $|\mathcal{U}_{Fair}|$ is equal to the number of components in all graphs in $\mathcal{G}^V$, we have that $dim(\mathcal{C}_{Fair}) \leq |\mathcal{U}_{Fair}|$. Therefore, $dim(\mathcal{C}_{Fair}) = |\mathcal{U}_{Fair}|$.

This concludes the sketch of proof of Theorem 6. □

Example 2. *Consider the degree centrality* c^D *from Example 1. Since adding an edge* $\{v, u\}$ *increases the centrality of both* v *and* u *by 1, then* c^D *satisfies Fairness. Consequently, we know from Theorem 6 that* c^D *is a linear combination of unanimity centralities from* $\mathcal{U}_{Fair}$*. Let us generate* c^D *using such a combination. To this end, consider* $c^{\langle \{v,u\}, \{\{v,u\}\} \rangle} \in \mathcal{U}_{Fair}$ *for some arbitrary pair,* $v, u \in V, v \neq u$*. According to* $c^{\langle \{v,u\}, \{\{v,u\}\} \rangle}$*, the centrality of* v *and* u *equals 1 if the edge* $\{v, u\}$ *belongs to the graph, otherwise the centrality of* v *and* u *equals 0. Summing over all such pairs, we get the degree centrality:*

$$c^D = \sum_{v, u \in V : v \neq u} c^{\langle \{v,u\}, \{\{v,u\}\} \rangle}.$$

Lemma 7. *Every separable game-theoretic centrality from* $\mathcal{GTC}_+$ *satisfies Fairness.*

We are ready to present our main result of this section. In the following theorem we state that the class of separable positive-semivalue based GTCs is characterized by Fairness.

Theorem 8. *For every positive semivalue,* $\varphi \in \mathcal{SV}_+$*,*

$$\mathcal{C}_{Fair} = [\mathcal{SGTC}_\varphi] = [\mathcal{SGTC}_+].$$

Sketch of Proof. Since $\varphi \in \mathcal{SV}_+$, then $\mathcal{SGTC}_\varphi \subseteq \mathcal{SGTC}_+$. Thus, based on Lemma 7: $[\mathcal{SGTC}_\varphi] \subseteq [\mathcal{SGTC}_+] \subseteq \mathcal{C}_{Fair}$. It remains to prove that $\mathcal{C}_{Fair} \subseteq [\mathcal{SGTC}_\varphi]$. In other words, we need to prove that every $c \in \mathcal{C}_{Fair}$ can be generated by $[(r, \varphi)]$ for some separable representation function r.

Note that $\mathcal{SGTC}$ is closed under addition and scalar multiplication (because if $r \in \mathcal{R}^V$ and $r' \in \mathcal{R}^V$ are separable, then $(r + r') \in \mathcal{R}^V$ and $(c \cdot r) \in \mathcal{R}^V$ are also separable). Thus, based on Lemma 4, it suffices to show that every centrality from $\mathcal{U}_{Fair}$ can be generated by $[(r, \varphi)]$ for some separable r. To this end, let $\varphi^\beta \in SV_+$ be an arbitrary positive semivalue based on some β, and let $c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{Fair}$. Now consider a representation function, r^*, defined as follows:

$$f^{r^*}_G(S) = \begin{cases} \frac{1}{\beta^*(|U|) + \beta^*(|U|-1)} & \text{if } S = U, E^\dagger \subseteq E, \\ \frac{\beta^*(|U|)}{(\beta^*(|U|) + \beta^*(|U|-1))\beta^*(|V|-1)} & \text{if } S = V, E^\dagger \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

First, we argue that r^* is separable, i.e., that the value of $f^{r^*}_G(S)$ depends solely on $G[S]$ and $G[V \setminus S]$. Since $c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{Fair}$, then based on Equation (3), if $S = U$ then S induces a component in $(V, E^\dagger)$, i.e., $S \in \mathcal{K}((V, E^\dagger))$. This implies that there are no edges between S and $V \setminus S$, meaning that no such edge can influence $f^{r^*}_G(S)$. Next, if $S = V$, then, by definition $f^{r^*}_G(S)$ depends solely on S. Finally, for $S \notin \{U, V\}$ we have $f^{r^*}_G(S) = 0$ for every graph $G \in \mathcal{G}$. Thus, r^* is separable.

It remains to show that $[(r^*, \varphi)] = c^{\langle U, E^\dagger \rangle}$. Let $G = (V, E)$ be an arbitrary graph, and let $v \in V$. Now if $E^\dagger \not\subseteq E$, then from the definition of unanimity we have: $c^{\langle U, E^\dagger \rangle}_v(G) = 0$, and from the definition of $f^{r^*}_G$ we have:

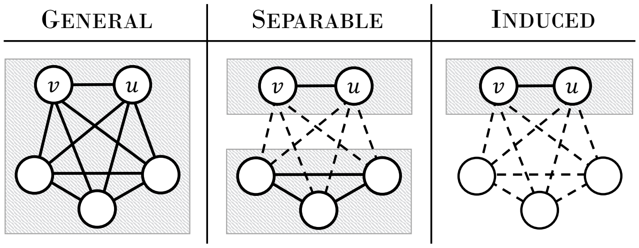

Figure 1: Given a complete graph of 5 nodes, the figure illustrates how the value of $\{v, u\}$ is computed under a general, a separable, and an induced representation function. The gray boxes contain the edges that affect the value of $\{v, u\}$; all remaining edges (i.e., the dashed ones) do not. Since there are 2^{10} possible graphs consisting of 5 nodes each, $\{v, u\}$ may have at most 2^{10} distinct values under a general representation function, r. In contrast, if r is separable, then $\{v, u\}$ may have at most 2^4 distinct values, and if r is induced, then $\{v, u\}$ may have at most 2^1 distinct values.

$f_G^{r^*}(S) = 0, \forall S \subseteq V$, implying that $\varphi_v(f_G^{r^*}) = 0$. Thus, we have: $c_v^{\langle U,E^\dagger\rangle}(G) = [(r^*, \varphi)]_v(G) = 0$. On the other hand, if $E^\dagger \subseteq E$, then we consider two cases: $v \in U$ and $v \notin U$. As for the first case, from the definition of unanimity, we have $c_v^{\langle U,E^\dagger\rangle}(G) = 1$. Furthermore, v has only two non-zero marginal contributions in Equation (1): one to coalition $S = U \setminus \{v\}$ and the other to coalition $S = V \setminus \{v\}$, implying that $\varphi_v^\beta(f_G^{r^*}) = 1$. Thus, we have: $c_v^{\langle U,E^\dagger\rangle}(G) = [(r^*, \varphi)]_v(G) = 1$. As for the second case, where $v \notin U$, one can verify that $c_v^{\langle U,E^\dagger\rangle}(G) = [(r^*, \varphi)]_v(G) = 0$. We have shown that in all cases we have: $c_v^{\langle U,E^\dagger\rangle}(G) = [(r^*, \varphi)]_v(G)$ for an arbitrary $G = (V, E) \in \mathcal{G}$ and an arbitrary $v \in V$. This concludes the sketch of the proof of Theorem 8. □

We end this section with an example showing how the degree centrality, c^D, can be generated from a separable GTC.

Example 3. *The degree centrality satisfies Fairness, because $\Delta_v^{c^D}(\{v, u\}, (V, E)) = 1$ for every $v \in V$ and every $\{v, u\} \notin E$. Therefore, we know from Theorem 8 that there exists a separable game-theoretic centrality index that generates c^D. Let us identify such a separable index. Note that the one used in Example 1 is not separable, because $f_{G_1}^r(\{v\}) = 0$ and $f_{G_2}^r(\{v\}) = 1$, while separability requires that $f_{G_1}^r(\{v\}) = f_{G_2}^r(\{v\})$. Instead, consider the index $[(r^D, \varphi^{Shapley})]$, where $\varphi^{Shapley}$ is the Shapley value, and r^D is defined as: $f_G^{r^D}(S) = 2 \cdot |\{\{v, u\} \in E : v, u \in S\}|$. Given this r^D, we show that $[(r^D, \varphi^{Shapley})] = c^D$.*

To this end, we will use the four widely-known axioms that define the Shapley value, namely: Additivity, Null-player, Symmetry *and* Efficiency.[1] *First of all, observe that: $f_G^{r^D}(S) = \sum_{e \in E} f_{(V,\{e\})}^{r^D}(S), \forall S \subseteq V$. Thus, based on the* Additivity *axiom, we have:*

$$\varphi_v^{Shapley}(f_G^{r^D}) = \sum_{e \in E} \varphi_v^{Shapley}(f_{(V,\{e\})}^{r^D}). \quad (4)$$

This allows us to focus our analysis on a single-edge graph, $(V, \{e\})$. Let us focus on $G^* = (V, \{\{v_1, v_2\}\})$. Here, it is clear from the definition of $f_{G^*}^{r^D}$ that the only two players with non-zero marginal contributions are v_1 and v_2. Thus, based on the Null-player *axiom: $\varphi_u^{Shapley}(f_{G^*}^{r^D}) = 0, \forall u \in V \setminus \{v_1, v_2\}$. As for v_1 and v_2, since they are symmetric, then based on the* Symmetry *axiom, we have: $\varphi_{v_1}^{Shapley}(f_{G^*}^{r^D}) = \varphi_{v_2}^{Shapley}(f_{G^*}^{r^D})$. Finally, since $f_{G^*}^{r^D}(V) = 2$, then based on the* Efficiency *axiom, we have: $\sum_{v \in V} \varphi_v^{Shapley}(f_{G^*}^{r^D}) = 2$. We have shown that the payoffs of all nodes in G^* add up to 2, and that v_1 and v_2 have equal payoffs, whereas the remaining nodes have zero payoff each, implying that $\varphi_{v_1}^{Shapley}(f_{G^*}^{r^D}) = \varphi_{v_2}^{Shapley}(f_{G^*}^{r^D}) = 1$. This as well as Equation* (4) *imply that every edge in G increases the payoff of each of its ends by 1, which is precisely what degree centrality does. Thus, $[(r^D, \varphi^{Shapley})] = c^D$.*

Note that r^D in Example 3 depends solely on $G[S]$. Such representation functions are the subject of the next section.

Induced Game-Theoretic Centralities

In this section, we define a subclass of separable GTCs which we call *induced* game-theoretic centralities.

Definition 3. *(Induced GTC ($\mathcal{IGTC}$)) A representation function, r, is* induced *if for every coalition $S \subseteq V$ and every two graphs $G, G' \in \mathcal{G}^V$ such that $G[S] = G'[S]$ it holds that $f_G^r(S) = f_{G'}^r(S)$. A GTC, (r, φ), is induced if r is induced. The set of all induced game-theoretic centralities is denoted by $\mathcal{IGTC}$.*

In words, a GTC is induced if the value of a coalition S in the representation function depends solely on the subgraph induced by S in G. Thus, every induced GTC is separable.

Given a complete graph of 5 nodes, Figure 1 illustrates the edges that affect the value of $\{v, u\}$ under a general, a separable, and an induced representation function.

To characterize the class of induced GTCs we introduce a new property that we call *Edge Balanced Contributions*.

Edge Balanced Contributions: *For every $G = (V, E)$, and every $e = \{v, \tilde{v}\}, e' = \{u, \tilde{u}\}$, $e, e' \notin E$, adding e' affects the difference in centrality of v caused by the addition of e in the same way that adding e affects the difference in centrality of u caused by the addition of e'. More formally:*

$$\Delta_v^c(e, (V, E \cup \{e'\})) - \Delta_v^c, (V, E)) = \Delta_u^c(e', (V, E \cup \{e\})) - \Delta_u^c(e', (V, E)). \quad (5)$$

Given a set of nodes, V, the class of all centrality indices satisfying Edge Balanced Contributions will be denoted by $\mathcal{C}_{EBC}^V$, or just $\mathcal{C}_{EBC}$ when V is clear from the context.

The new property is a modification of the *Balanced Contributions* property, introduced by Myerson in the context of

[1]For more on the various axiomatizations of the Shapley value, see, e.g., the work by Maschler et al. (2013).

coalitional games (Myerson 1980). Balanced Contributions states that removing player i from the game affects the payoff of player j in the same way as removing player j affects the payoff of player i. If we associate with removing of an edge the effect this removal has on both adjacent nodes, then Edge Balanced Contribution is an edge counterpart of Balanced Contributions.

Note that Edge Balanced Contributions implies Fairness. In particular, by setting $u = \widetilde{v}$ and $\widetilde{u} = v$, we have $e' = e$, and we get $\Delta^c_v(e, (V, E \cup \{e'\})) = 0$. Then, Equation (5) simplifies to: $-\Delta^c_v(\{v, u\}, (V, E)) = -\Delta^c_u(\{v, u\}, (V, E))$ for every $\{v, u\} \notin E$, which is equivalent to Fairness.

Now, let $\mathcal{K}_s(G)$ be the set of nodes that induce single-node components in G, i.e., $\mathcal{K}_s(G) = \{v \in V : \{v\} \in \mathcal{K}(G)\}$, where the "$s$" in $\mathcal{K}_s$ stands for "*single-node*". The following lemma states that any centrality index in $\mathcal{C}_{EBC}$ can be uniquely characterized by specifying (1) the centrality of every single-node component in G; and (2) the sum of node-centralities in every other component in G.

Lemma 9. *For every function, $g : 2^V \times \mathcal{G}^V \to \mathbb{R}$, there exists at most one centrality index, $c \in \mathcal{C}^V_{EBC}$ that satisfies $\sum_{v \in V \setminus \mathcal{K}_s(G)} c_v(G) = g(V \setminus \mathcal{K}_s(G), G)$ and $c_v(G) = g(\{v\}, G)$ for every $G \in \mathcal{G}^V$ and every $v \in \mathcal{K}_s(G)$.*

Building upon the above lemma, the following theorem identifies a basis of the class $\mathcal{C}_{EBC}$.

Theorem 10. *$\mathcal{C}_{EBC}$ is a vector space with the basis:*

$$\mathcal{U}^{EBC} = \left\{ c^{\langle \{v\}, E^\dagger \rangle} : v \in \mathcal{K}_s\left((V, E^\dagger)\right) \right\} \cup \left\{ c^{\langle U, E^\dagger \rangle} : U = V \setminus K_s\left((V, E^\dagger)\right) \right\}.$$

Sketch of Proof. Since $\Delta^c_v(e, G)$ is a linear function, then if c and c' satisfy Edge Balanced Contributions, then $c + c'$ and $k \cdot c$, for all $k \in \mathbb{R}$ also satisfy Edge Balanced Contributions. Thus, $\mathcal{C}_{EBC}$ is a vector space. To prove that $\mathcal{U}_{EBC}$ forms a basis of $\mathcal{C}_{EBC}$ we use a reasoning similar to that of the proof of Theorem 6. In particular, we first show that $\mathcal{U}_{EBC} \subseteq \mathcal{C}_{EBC}$, then that $\mathcal{U}_{EBC}$ is linearly independent, and, finally, that $|\mathcal{U}_{EBC}|$ is equal to the dimension of $\mathcal{C}_{EBC}$.

As for the first step, let $c = c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{EBC}$. Consider the value $x = \Delta^c_v(e, (V, E \cup \{e'\})) - \Delta^c_v(e, (V, E))$ for $e = \{v, \widetilde{v}\}, e' = \{u, \widetilde{u}\}$. This value is not equal to zero only if $v \in U$ and $e, e' \in E^\dagger$ and E contains the other edges from $E^\dagger$. In such a case, $x = 1$. However, from the definition of $\mathcal{U}_{EBC}$, if $v \in U$ and $\{v, \widetilde{v}\} \in E^\dagger$, then $U = V \setminus \mathcal{K}_s(G)$. Therefore, $u \in U$ and $\Delta^c_u(e', (V, E \cup \{e\})) - \Delta^c_u(e', (V, E)) = x$, i.e., Edge Balanced Contributions is satisfied. We proved that an arbitrary $c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{EBC}$ satisfies Edge Balanced Contributions. Hence, $\mathcal{U}_{EBC} \subseteq \mathcal{C}_{EBC}$.

As for the second step, it suffices to note that the linear independence of $\mathcal{U}_{EBC}$ is implied by Lemma 2.

As for the third step, let us denote by $dim(\mathcal{C}_{EBC})$ the dimension of $\mathcal{C}_{EBC}$. Linear independence implies that $dim(\mathcal{C}_{EBC}) \geq |\mathcal{U}_{EBC}|$. At the same time, from Lemma 9 as well as the fact that $|\mathcal{U}_{EBC}|$ equals the number of single-node components in all graphs from $\mathcal{G}$ plus the number of graphs from $\mathcal{G}$ with at least one edge, we have that $dim(\mathcal{C}_{EBC}) \leq |\mathcal{U}_{EBC}|$. Consequently, $dim(\mathcal{C}_{EBC}) = |\mathcal{U}_{EBC}|$. This concludes the sketch of proof. □

Lemma 11. *Every induced game-theoretic centrality index in $\mathcal{GTC}_+$ satisfies Edge Balanced Contributions.*

Finally, as the main result of this section, we prove that the class of induced positive semivalue-based GTCs is characterized by the property of Edge Balanced Contributions.

Theorem 12. *For every positive semivalue, $\varphi \in \mathcal{SV}_+$,*

$$\mathcal{C}_{EBC} = [\mathcal{IGTC}_\varphi] = [\mathcal{IGTC}_+].$$

Sketch of Proof. Since $\varphi \in \mathcal{SV}_+$, then $\mathcal{IGTC}_\varphi \subseteq \mathcal{IGTC}_+$. Thus, based on Lemma 11 we have that $[\mathcal{IGTC}_\varphi] \subseteq [\mathcal{IGTC}_+] \subseteq \mathcal{C}_{EBC}$. It remains to prove that $\mathcal{C}_{EBC} \subseteq [\mathcal{IGTC}_\varphi]$. In other words, we need to prove that every $c \in \mathcal{C}_{EBC}$ can be generated by $[(r, \varphi)]$ for some induced representation function, r. Note that $\mathcal{IGTC}$ is closed under addition and scalar multiplication (because if r and r' are induced, then $r + r'$ and $k \cdot r$ for every $k \in \mathbb{R}$ are also induced). Thus, based on Lemma 4, it suffices to show that every centrality index from $\mathcal{U}_{EBC}$ can be generated by $[(r, \varphi)]$ for some induced r.

To this end, we will first show that any unanimity centrality index, $c^{\langle U, E^\dagger \rangle}$, such that every edge in $E^\dagger$ is between nodes from U (i.e., $\{v, u\} \in E^\dagger \to v, u \in U$) can be generated by an induced GTC from $\mathcal{GTC}_+$. Let $\varphi^\beta \in SV_+$ be a positive semivalue based on an arbitrary β, and let $r^{\langle U, E^\dagger \rangle}$ be a representation function defined as follows:

$$f_G^{r^{\langle U,E^\dagger \rangle}}(S) = \begin{cases} \left(\sum_{k=|U|}^{|V|} \binom{|V|-|U|}{k-|U|} \beta^*(k-1) \right)^{-1} & \text{if } U \subseteq S, E^\dagger \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

Since all edges from $E^\dagger$ are between nodes from U, and since $U \subseteq S$, then $f_G^{r^{\langle U,E^\dagger \rangle}}(S)$ depends solely on $G[S]$, which means that $r^{\langle U, E^\dagger \rangle}$ is induced.

We can show now that $[(r^{\langle U, E^\dagger \rangle}, \varphi)] = c^{\langle U, E^\dagger \rangle}$. If $E^\dagger \not\subseteq E$, both centralities assign 0 to all nodes. Assume then that $E^\dagger \subseteq E$. If $v \notin U$, then v has zero marginal contribution to every coalition and $[(r^{\langle U, E^\dagger \rangle}, \varphi)]_v(G) = 0 = c_v^{\langle U, E^\dagger \rangle}(G)$. If $v \in U$, then v has non-zero marginal contribution to coalition S if and only if $U \setminus \{v\} \subseteq S$. Using Equation (1), we obtain: $[(r^{\langle U, E^\dagger \rangle}, \varphi)]_v(G) = 1 = c_v^{\langle U, E^\dagger \rangle}(G)$. Therefore, $[(r^{\langle U, E^\dagger \rangle}, \varphi)] = c^{\langle U, E^\dagger \rangle}$. Finally, we are ready to show that every unanimity centrality from $\mathcal{U}_{EBC}$ can be generated by some separable $[(r, \varphi)]$. To this end, if $c^{\langle U, E^\dagger \rangle} \in \mathcal{U}_{EBC}$, then either $U = V \setminus \mathcal{K}_s((V, E^\dagger))$, or $U = \{v\} : v \in \mathcal{K}_s((V, E^\dagger))$. In the first case, we already proved that $[(r^{\langle U, E^\dagger \rangle}, \varphi)] = c^{\langle U, E^\dagger \rangle}$. In the second case, to generate $c^{\langle \{v\}, E^\dagger \rangle}$, we use the centrality $[(r^{\langle S \cup \{v\}, E^\dagger \rangle} - r^{\langle S, E^\dagger \rangle}, \varphi)]$, with $S = V \setminus \mathcal{K}_s((V, E^\dagger))$. This concludes the sketch of the proof of Theorem 12. □

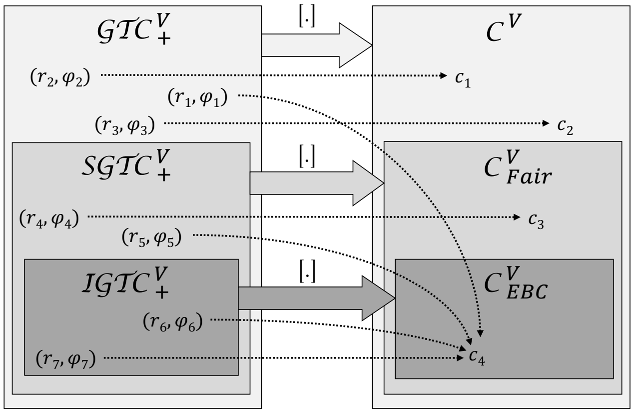

Figure 2: An illustration summarizing our results. Every small (dotted) arrow connects a game-theoretic centrality, (r_i, φ_i), to the corresponding centrality $c_i = [(r_i, \varphi_i)]$. Every large arrow connects some subclass of GTCs, $\mathcal{I}$, to $[\mathcal{I}]$ – the class of centrality indices generated by $\mathcal{I}$. As we have shown in Theorem 1, $[\mathcal{GTC}_+]$ encompasses all centralities, i.e., for every centrality, c, there is an incoming arrow from some game-theoretic centrality $(r, \varphi) \in \mathcal{GTC}_+$. In Theorem 8, we proved that all arrows from *Separable* GTCs go into centralities that satisfy *Fairness*, and that for every centrality satisfying *Fairness* there exists an incoming arrow from some Separable GTC. In Theorem 12, we showed that all arrows from *Induced* GTCs go into centralities that satisfies *Edge Balanced Contributions*, and that for every centrality satisfying *Edge Balanced Contributions* there exists an incoming arrow from some induced game-theoretic centrality. Note that it is possible that a non-separable game-theoretic centrality generates a centrality satisfying *Fairness* or *Edge Balanced Contributions* (e.g., $[(r_1, \varphi_1)] = c_4$).

Conclusions

We studied an axiomatic characterization of game-theoretic centralities. Our results are summarized in Figure 2. We showed that, while all centralities can be obtained by the game-theoretic approach, some natural classes of game-theoretic centralities are characterized by *Fairness* and its strengthening – *Edge Balanced Contributions*. This suggests that the game-theoretic approach is a good choice when the nodes are assessed based on some property that agrees with Fairness (a good example of such a property is "connectivity", as in the work by Skibski et al. 2016). Although this finding does not give us the complete answer to the question "which centrality is better for a specific application", it brings us closer to it.

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