A Theoretical Framework of the Graph Shift Algorithm

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Abstract

Since no theoretical foundations for proving the convergence of Graph Shift Algorithm have been reported, we provide a generic framework consisting of three key GS components to fit the Zangwill's convergence theorem. We show that the sequence set generated by the GS procedures always terminates at a local maximum, or at worst, contains a subsequence which converges to a local maximum of the similarity measure function. What is more, a theoretical framework is proposed to apply our proof to a more general case.

Introduction

In treating the traditional dense subgraph discovery problem, Pavan et al (Pavan and Pelillo 2007) put forward a framework called Dominant Sets and Pairwise Clustering (DSPC) Algorithm, treating the dense subgraph discovery problem as a constrained optimization problem. Liu et al (Liu and Yan 2010) further modified it, and proposed a Graph Shift (GS) Algorithm for the graph mode seeking.

The GS Algorithm employs an iterative strategy to find the local maximum of its constraint objective function. However, none of the existing work has proven the convergence of the GS Algorithm. It is certainly crucial to have a theoretical analysis about the GS Algorithm's convergence before we confidently utilize it.

Here we break down the GS Algorithm into three key components and prove them all perfectly match with the conditions of Zangwill's convergence theorem (Zangwill 1969). Thus the convergence is come out under the guidance of Zangwill's theorem. What is more, a general framework for all the GS-type algorithm is then brought about.

It is a brevity poster. For more details, please refer to [1].

Relations

The Graph Shift (GS) Algorithm consists of three components, all to identify itself from others.

1), Simplex of generated sequence set The candidate solution sequence set $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ generated by each iteration lies in $\Delta^n = \{\boldsymbol{x} \in R^n : \boldsymbol{x}_i \geq 0 \text{ and } |\boldsymbol{x}|_1 = 1\}$, i.e., the standard n-simplex of R^n in any step k ($k \leq m$);

2), Monotonic and continuous objective function The objective function $g(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$ is both continuous and strictly increases during the iteration according to the Propositions 2-4 (See Section 3);

3), Closed mapping The iteration mapping $T_m = B^{m_k} \circ C$ (B^{m_k} is Replicator Dynamics and C is Neighborhood Expansion) is closed during each procedure in accordance to Propositions 5-6 (see Section 4) at all points of the generated sequence set(including the solution set Γ).

Accordingly, the Zangwill's convergence theorem (Zangwill 1969) is described below.

Theorem 1. *Given an algorithm on X, $\boldsymbol{x}^{(0)} \in X$, assume the sequence $\{\boldsymbol{x}^{(k)}\}_{k=1}^{\infty}$ is generated which satisfies*

$$\boldsymbol{x}^{(k+1)} \in \boldsymbol{A}(\boldsymbol{x}^{(k)}) \tag{1}$$

For a given solution set $\Gamma \subset X$ of an algorithm, if the following three properties holds:

Compact *The sequence set $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty} \subset S$ for $S \subset X$ is a compact set.*

Decreasing *There is a continuous function Z on X such that*

1) if $\boldsymbol{x} \notin \Gamma$, then $Z(\boldsymbol{y}) < Z(\boldsymbol{x})$ for all $\boldsymbol{y} \in \boldsymbol{A}(\boldsymbol{x})$.

2) if $\boldsymbol{x} \in \Gamma$, then $Z(\boldsymbol{y}) \leq Z(\boldsymbol{x})$ for all $\boldsymbol{y} \in \boldsymbol{A}(\boldsymbol{x})$.

Closed *The mapping $\boldsymbol{A}$ is closed at all points of $X \backslash \Gamma$.*

Then either the algorithm stops at the point where a solution is identified or there exists such a k so that for all $k + j$ ($j \geq 1$) there is a convergent subsequence of $\{\boldsymbol{x}^{(i_k)}\}_{k=0}^{\infty}$ in the solution set Γ.

The three conditions of Zangwill convergence theorem share similarities with the GS Algorithm's properties as shown in Table 1.

Convergence of the GS Algorithm

Several propositions for the three GS Algorithm's components are presented first to facilitate the convergence proof. Detail proofs of these propositions refer to the complementary materials.

Proposition 1 discusses the generated set's property.

[1] https://sites.google.com/site/xhfanlee/home/a-convergence-theorem

GS Algorithm		Zangwill's theorem
$Simplex$	$\sim$	$Compact$
$Monotonic$	$\sim$	$Decreasing$
$Closed$	$\sim$	$Closed$

Table 1: GS Algorithm and Zangwill's Theorem's similarities

Proposition 1. *The sequence set* $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty} \subset S$ *generated by the mapping* $T_m = B^{m_k} \circ C$ *is a compact set.*

Propositions 2-4 discuss the monotonicity of $g(\boldsymbol{x})$ under the mapping $T_m = B^{m_k} \circ C$.

Proposition 2. *The objective function* $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ *strictly increases along the Replicator Dynamics* (B^{m_k}) *when* $\boldsymbol{x} \in X/\Gamma$.

Proposition 3. *The objective function* $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ *strictly increases along the neighborhood expansion* (C) *operation.*

Proposition 4. $g(x) = \boldsymbol{x}^T A \boldsymbol{x}$ *is a function both continuous and strictly increasing during the mapping* $T_m = B^{m_k} \circ C$ *when* $\boldsymbol{x} \in X/\Gamma$*, but just increasing if* $\boldsymbol{x} \in \Gamma$.

Propositions 5-6 validate the closed property of the GS Algorithm's mapping T_m.

Lemma 1. *Given two continuous functions:* $f : I \to J(\subset \boldsymbol{r}), g : J \to \boldsymbol{R}$*, the composition* $g \circ f : I \to \boldsymbol{R}, x \mapsto g(f(x))$ *is continuous.*

Proposition 5. *The mapping* C *is closed on all points of* $X \backslash \Gamma$.

Proposition 6. *The mapping* $T_m = B^{m_k} \circ C$ *is closed on* X/Γ.

With the above preparations, we have the following Theorem 2.

Theorem 2. *Let* $A = (a_{ij})^{(n \times n)}$ *be a similarity matrix with diagonal values 0,* T_m *be the GS Algorithm's mapping,* Γ *be the solution set, and* $\boldsymbol{x}^{(0)}$ *be an arbitrary initial starting point, then either the iteration sequence* $\{\boldsymbol{x}^{(r)}\}$ $(r = 1, 2, \ldots)$ *terminates at a point* $\boldsymbol{x}^*$ *in the solution set* Γ *or there is a subsequence converging to a point in* Γ.

Proof. Taking $\hat{g}(\boldsymbol{x}) = -g(\boldsymbol{x}) = -\boldsymbol{x}^T A \boldsymbol{x}$ as the continuous function Z and T_m as the algorithm mapping A in Theorem 1. Proposition 1 shows that the sequence set $\{\boldsymbol{x}^{(k)}\}_{k=1}^{\infty}$ generated by T_m is a compact set. $\hat{g}(\boldsymbol{x})$ is continuous and strict decreasing as the continuity and strict increasing characteristic of $g(\boldsymbol{x})$ in the trajectory of T_m are proven by Proposition 4. Proposition 6 asserts T_m is closed on x/Γ (Γ is the solution set). According to the Zangwill's convergence theory, Theorem 2 holds as all three properties are satisfied. □

GS-type Algorithm convergence framework

Here, we further propose a framework for the convergence proof of GS-type algorithms.

Def. 1. *An algorithm is a GS-type algorithm if and only if it satisfies the conditions on three key components: simplex of generated sequence set, monotonic and continuous objective function and closed mapping.*

The proof of Theorem 2 provides us a guideline on GS-type algorithms' proving. (1) Breaking down the GS-type algorithm into three parts: generated set, objective function and mapping. (2) Check if these three parts all satisfy the corresponding requirements. Any part unsatisfied is regarded as unsuitable for applying this framework, otherwise it is convergent under our framework.

Figure 1 shows the framework to prove the convergence of GS-type algorithms step by step.

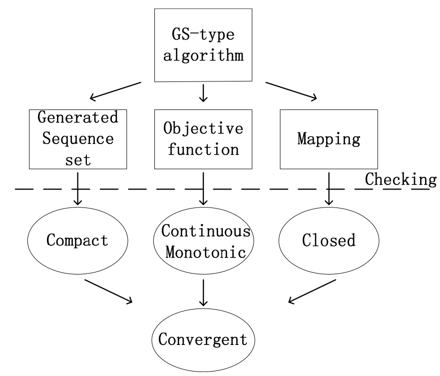

Figure 1: Proving framework for GS-type algorithm convergence

Based on the proposed framework, we can prove that other GS-type algorithms such as the DSPC Algorithm (Pavan and Pelillo 2007) are applicable for the above process and they are convergent.

Conclusions and Future work

We propose a generic theoretical framework to prove the convergence of the Graph Shift (GS) Algorithm. The GS Algorithm's extracted three key components are mapped to the Zangwill convergence theorem's three key conditions. Consequently, the convergence of the GS Algorithm is proved by applying the Zangwill convergence theorem. Also, we extend our work to the GS-type algorithm and build up a solid theoretical framework to this general GS-type algorithms.

In the future, Banach's contraction theory and optimization methods on nonconvex mathematical program are being considered in our problem so as to suit in a more general case.

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