M-Unit EigenAnt: An Ant Algorithm to Find the *M* Best Solutions

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Abstract

In this paper, we shed light on how powerful congestion control based on local interactions may be obtained. We show how ants can use repellent pheromones and incorporate the effect of crowding to avoid traffic congestion on the optimal path. Based on these interactions, we propose an ant algorithm, the M-unit EigenAnt algorithm, that leads to the selection of the M shortest paths. The ratio of selection of each of these paths is also optimal and regulated by an optimal amount of pheromone on each of them. To the best of our knowledge, the M-unit EigenAnt algorithm is the first ant algorithm that explicitly ensures the selection of the M shortest paths and regulates the amount of pheromone on them such that it is asymptotically optimal. Infact, it is in contrast with most ant algorithms that aim to discover just a single best path. We provide its convergence analysis and show that the steady state distribution of pheromone aligns with the eigenvectors of the cost matrix, and thus is related to its measure of quality. We also provide analysis to show that this property ensues even when the food is moved or path lengths change during foraging. We show that this behavior is robust in the presence of fluctuations and quickly reflects the change in the M optimal solutions. This makes it suitable for not only distributed applications but also dynamic ones as well. Finally, we provide simulation results for the convergence to the optimal solution under different initial biases, dynamism in lengths of paths, and discovery of new paths.

Nature is ubiquitous with networks that efficiently control the flow of resources and information, yet the efficient distribution by decentralized, individual decisions is still an open problem in many real world systems with limited resources. The foraging networks of terrestrial animal societies, especially those of certain ant species, provide unrivaled opportunities to quantify both congestion control on the optimal path and the optimal resource usage. Ants are known to use 'stigmergy' (indirect communication via chemical cues) and follow simple rules to avoid traffic jams (Jaffe 1980), avoid congestion on optimal paths, and use alternative sub-optimal paths to obtain a higher efficiency globally (Dussutour et al. 2004; Couzin and Franks 2003). For these reasons, they are

ideal subjects with which to test mathematical models that link the behavior of small components (in this instance, individual ants and their individual decision making, interactions and coordination) to the overall efficiency of the dynamic structures they generate.

Ants can drop and sense pheromone on a path that they travel on. Amongst a set of alternative paths, an ant chooses a path that has a higher pheromone concentration. Thus, a path that has been traveled more by preceding ants has a higher concentration of pheromone and has a higher probability of being chosen by succeeding ants (Deneubourg et al. 1990; Bonabeau, Dorigo, and Theraulaz 1999). Given two alternative paths to the food source, with equal pheromone concentration on them, half of the ants would choose one path and the other half would choose the other path. The ants that chose the shorter path would reach the food source quicker. These ants will detect a pheromone trail only on the shorter path for the return journey. This is because ants that randomly chose the longer path would not have reached the food source, and thus, there is no pheromone concentration on the longer path near the food source. Therefore, the returning ants will choose the shorter path, further increasing the pheromone concentration along the shorter path. After a while, ants that took the longer path reach the food source. For their return, they choose the shorter path since it has a higher concentration of pheromone. Similar actions eventually lead to all ants using the shorter path. What is remarkable is the fact that the optimization (selection of the shortest path), is achieved without explicit communication about the merits of one path over another, since an ant having chosen one path has no idea of the length of the other path, or any centralization.

What is also remarkable, and not completely understood to date, is the fact that ants are able to regulate the density of ants on the optimal path to avoid traffic jams, increase the global efficiency of the colony, and also forage from multiple food sources (Jaffe 1980; Jackson et al. 2006). This is possible again through coordination based on the adjusting of amount of pheromone dropped on paths. The regulation of traffic flow and optimal resource usage is an extremely important optimization problem that arises in all routing applications, resource allocation, and congestion avoidance. In these situations, it is required that several short paths (or good solutions) be used.

In this paper, we shed light on how powerful congestion control based on local interactions may be obtained. We show how ants can use repellent pheromones and incorporate the effect of crowding to avoid traffic congestion on the optimal path. Based on these interactions, we propose an ant algorithm to be named, the M-unit EigenAnt algorithm, that leads to the selection of the M shortest paths. The distribution of number of ants selecting each of these paths is optimal because it is proportional to its shortness (or any measure of quality). We provide convergence analysis to prove that the steady state distribution of pheromone on each path is related to its measure of quality.

We begin with describing the biological motivations behind our pheromone update rule. We proceed to describe the algorithm and provide its convergence analysis and properties. We also present simulation results to verify the convergence properties of the algorithm under different conditions that lead to stagnation and premature convergence to non-optimal solutions. Finally we conclude with some observations.

Biological Motivations

Let us assume that there are N alternative paths available to a colony of ants. We denote the distribution of pheromone concentration by the vector C. The i-th component of C, denoted by C_i, represents the amount of pheromone on the i-th path. We denote the total pheromone concentration at the food source by V, that is, $V = \sum_{i=1}^{N} C_i$. The length of the i-th path is denoted by l_i. It is known that the likelihood of an ant picking the i-th path is a function of $\frac{C_i}{V}$ (Deneubourg et al. 1990; Sumpter and Beekman 2003), where $V = \sum_{i=1}^{N} C_i$. Once an ant chooses the i-th path, it drops d_i amount of pheromone on it. We use D to represent a diagonal matrix whose elements are the diagonal elements d_i. This matrix represents the cost matrix of the problem.

Many Ant Colony Optimization (ACO) algorithms are based only this positive reinforcement based selection of the shortest path model. However, if there is more initial pheromone concentration on the longer path, unlike the case described in the previous section, it has been proven that only positive reinforcement cannot lead to the selection of shortest path. Consider the situation when a shorter path is discovered after the discovery of a long path. In that situation, the longer path would have a large build-up of pheromone before the shorter path is discovered on account of ants having used it. Thus, it would continue to be preferred over the recently discovered shorter path (Shah et al. 2008). The quick returns on the shorter path can revert only some of the initial bias on the longer path, beyond which a longer path will continue to get reinforced despite a shorter path having been discovered (Shah et al. 2010). Thus, beyond this limit, it becomes necessary to introduce some mechanism to bring the system out of the premature convergence to a non-optimal path. Not only should any ACO algorithm be able to revert the initial bias on sub-optimal paths, but also it should be *Plastic* enough to allow the movement of trail to a newly discovered optimal solution. The efficiency of these mechanisms determines the quality of so-

lution converged to.

Some of these mechanisms include, introducing a minimum amount of pheromone that a path will always have, hence avoiding situations in which the probability of choosing of any path (which could be optimal) becomes zero (Ant Colony System (ACS) (Dorigo and Gambardella 1997)). Some algorithms like MaxMin Ant System (Stützle and Hoos 2000) fix the maximum amount of pheromone that can be dropped on a path. These modifications cause ACS and MMAS to be amongst one of the best performing heuristics (Dorigo and Blum 2005) in practice, however, they do not ensure convergence to the optimal solution. Another commonly used strategy is to introduce negative feedback in the form of evaporation (Dorigo, Maniezzo, and Colorni 1996). Evaporation makes it difficult to maintain pheromone trails on longer paths and improves the solution quality. However, there are theoretical limits to the amount of initial bias that can be reverted using these approaches (Shah et al. 2010).

In this paper, we use a negative feedback mechanism that is based on the concept of 'repellent pheromones' dropped by ants in trail formation. Several biological studies (Robinson, Ratnieks, and Holcombe 2008; Jaffe 1980; Jackson et al. 2006; Dussutour et al. 2004) have reported a key role of repellent pheromones in governing trail formation. Jaffe (Jaffe 1980) reported a decrease in the recruitment effort at increasing ant densities on the trail. This produced more efficient recruitment and was attributed to the use of repellent pheromones. Despite the biological motivations, its use in ACO algorithms and applications has been mostly ignored. In this paper, we show the use of repellent pheromones in congestion control and selection of the M shortest paths. Note that repellent pheromone is dropped only on the chosen path and hence is different from an evaporative removal that affects all paths. We introduce a term $-\alpha \, C_i$ in the model, to incorporate the effect of repellent pheromone.

The amount of trail-pheromone is released in accordance with the quality of the food (Jaffe 1980) and the concentration of pheromone already present on the trail. In our model, we assume that the additional pheromone (d_i) deposited at the nest end of the i-th path, laid by a returning ant, is proportional to $f(l_i)$, where $f(\cdot)$ is any monotonically decreasing function. This term can also be justified in the context of the physiology of ant's pheromone glands (Bossert and Wilson 1963).

We term a set of visits by ants from the nest to the food source, and back, as an epoch. Putting all the assumptions together, we can describe the working of the model as follows: In an epoch, $\frac{C_i}{V}$ ants choose the i-th path. These ants deposit pheromone $\beta f(l_i)$ or βd_i, per unit length. Hence, the total pheromone deposited on the i-th path near the food source because of $\frac{C_i}{V}$ ants having chosen it, will be $\beta d_i \frac{C_i}{V}$. Negative feedback results in removal of αC_i pheromone from the amount deposited.

Analytically, we obtain the rule for change in the concentration of pheromone on the i-th path at the t-th epoch of the form:

$$C_i^{t+1} = C_i^t - \alpha C_i^t + \beta d_i \frac{C_i^t}{V^t}. \qquad (1)$$

There are several other microscopic interactions that come into play while recruitment, for example, communications may involve stroking, gasping, antenna movements, and streaking of chemicals. However, in this work, our aim is to show at a macroscopic level how the local interactions based on stigmergy can lead to congestion control and optimal traffic distribution and we ignore these and several other microscopic interactions. In the next section, we show how we can develop this model such that it leads to the selection of the M shortest paths asymptotically and also optimal distribution of pheromone on them.

M-Unit EigenAnt Algorithm

In this section, we present our M-unit EigenAnt algorithm[1]. The algorithm is parameterized by M that allows the maintaining of trail on the M best solutions, which is selected depending on the application at hand.

Since the goal is to lead to the selection of M paths, we use C to denote a pheromone matrix instead of a vector. Note that the use of a matrix may yield this algorithm implausible but the association of a vector with each path is a common practice in artificial ant algorithms. The order of the matrix C is $N \times M$ where N denotes the number of paths and M denotes the cardinality of the set of optimal solutions required. The alternative paths need not be disjoint paths and there may be shared edges between them. In the pheromone matrix C, a component $C_{i,j}$ denotes the pheromone on the i-th path while j is used as an index to differentiate between the M best solutions.

For the M-unit EigenAnt rule, the change in concentration at the t-th epoch for pheromone on the i-th path w.r.t the j-th winner, is determined as

$$\Delta C_{i,j}{}^t = -\alpha \sum_{k=1}^{j} C_{i,k}{}^t + \beta\, d_i\, \frac{C_{i,j}^t}{V_j^t} , \qquad (2)$$

where, $V_j^t = \sum_{p=1}^{N} C_{p,j}{}^t$. If a path i is the $j-th$ optimal solution, and $j \leq M$ then only $C_{i,j}$ will be non-zero and all other components of C_i will be zero asymptotically.

The total number of non-zero components of C at the steady state, is determined by the parameter M. In the next section, we show that for a particular choice of M, one can obtain the M most optimal solutions. The components of C corresponding to the remaining $N - M$ solutions will go to zero. Also, the pheromone distribution on the M optimal solutions will be inversely proportional to their cost, while the pheromone distribution on the remaining $N - M$ paths will go to zero.

Convergence Analysis

We begin with determining which path is asymptotically selected as the most optimal solution, that is, which i-th path is chosen for the corresponding index $j = 1$. For the system of equation (2), an equilibrium point is characterized by

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[1]The algorithm is motivated by Sanger's rule (Sanger 1989) and Oja's M-unit rule. Similarly, we name it the M-unit EigenAnt rule

$\langle \Delta C_{k,1} \rangle = 0$, for each path $k = 1,\ 2,\ \ldots,\ N$.

$$\langle \Delta C_{k,1} \rangle = -\alpha C_{k,1} + \beta d_k \frac{C_{k,1}}{V_1} = 0 . \qquad (3)$$

Representing $C_{k,1}$ where $k = 1,\ 2,\ \ldots,\ N$ by a variable x and using D to denote a diagonal matrix whose entries $d_i = f(l_i)$, we get,

$$(\alpha\, V_1)x = \beta D x . \qquad (4)$$

In other words, the pheromone concentration vector x is an eigenvector of the diagonal matrix D, and we may write

$$Dx = \lambda x , \qquad (5)$$

where, λ represents the eigenvalues.

Since D is a diagonal matrix, its eigenvectors are unit vectors, each corresponding to a distinct path. Hence, at steady state, the expected pheromone distribution is one in which all the deposited pheromone is concentrated on one path. Thus, corresponding to an index j there will only be one path i that will have a non-zero component $c_{i,j}$.

If the i-th eigenvalue is denoted by λ_i, then we note that

$$\lambda_i = d_i = \left(\frac{\alpha}{\beta} \sum_{k=1}^{N} C_{k,1} \right)$$

$$\Rightarrow \left(\sum_{k=1}^{N} C_{k,1} \right) = d_i . \frac{\beta}{\alpha} . \qquad (6)$$

Equation (6) indicates that the total pheromone deposited on the path is bounded at steady state. This is sensible in that as long as a food source is available, ant trails should continue getting reinforced. The total pheromone does not grow out of bound because of the steady state repellent pheromone amount deposition that counters it.

Any normalized eigenvector of D would satisfy equation (5), but we need to show that the only stable eigenvector would be the one corresponding to the largest eigenvalue. This corresponds to the shortest path because $d_i = f(l_i)$ is a monotonically decreasing function of l_i.

Let the eigenvector corresponding to eigenvalue λ_i be denoted by u^i, and suppose that the pheromone distribution is given by $x = u^i$, that is, the pheromone is concentrated on the i-th path. At steady state, we have

$$Du^i = \lambda_i u^i = \frac{\alpha}{\beta} V_1 u^i = \frac{\alpha}{\beta}(e^T u^i)u^i , \qquad (7)$$

where, $e^T = [1, 1, \ldots, 1]_N$.

Suppose that there is a perturbation of ε around the steady state value of x, that is, let x be perturbed to $x = u^i + \varepsilon$. Consider the value of

$$\langle V_1 (\Delta x + \Delta \varepsilon) \rangle = \langle V_1 \Delta \varepsilon \rangle = \beta D(x + \varepsilon) - \alpha V_1 (x + \varepsilon) . \qquad (8)$$

Inserting $x = u^i$, we obtain

$$\beta D(u^i + \varepsilon) - \alpha[e^T(u^i + \varepsilon)](u^i + \varepsilon). \qquad (9)$$

Simplifying the above, we get

$$[\beta Du^i\ -\ \alpha(e^T u^i)u^i]\ +\ \beta D\varepsilon\ -\ \alpha(e^T \varepsilon)u^i$$
$$-\ \alpha(e^T u^i)\varepsilon\ -\ \alpha(e^T \varepsilon)\varepsilon . \qquad (10)$$

The term within square parenthesis is zero due to (7). The last term, $\alpha(e^T\varepsilon)\varepsilon$, is $O(\varepsilon^2)$, and thus negligible. Thus, we can write (10) as,

$$\beta D\varepsilon - \alpha(e^T\varepsilon)u^i - \alpha(e^Tu^i)\varepsilon . \qquad (11)$$

We now examine the component of $\langle\Delta\varepsilon\rangle$ in the direction of another eigenvector, say u^j. We have $(u^j)^TV_1\langle\Delta\varepsilon\rangle$ as,

$$= (u^j)^T\beta D\varepsilon - \alpha(e^T\varepsilon)(u^j)^Tu^i - \alpha(e^Tu^i)(u^j)^T\varepsilon$$
$$= \varepsilon^TDu^j - \alpha\delta_{ij}(e^T\varepsilon) - d_i(u^j)^T\varepsilon$$
$$= (d_j - d_i)(u^j)^T\varepsilon - \alpha\delta_{ij}(e^T\varepsilon), \quad (12)$$

where, δ_{ij} is the Kronecker delta function.

Equation (12) indicates that a perturbation of $+\varepsilon$ will have a growing component in the direction of a different eigenvector u^j if $(d_j > d_i)$. Such an eigenvector always exists unless u^i corresponds to the largest d_i. In other words, x converges to the eigenvector corresponding to the largest eigenvalue; when this happens, the first term is zero or negative for all j, and the second term is negative. Hence, following any perturbation $+\varepsilon$, the pheromone vector changes in the opposite direction to restore x to its earlier value. Since, the largest eigenvalue corresponds to the shortest path, the first winner, corresponding to $j = 1$ will converge to the shortest path. Note that not only does the algorithm reach convergence to the optimal solution irrespective of the initial conditions on the pheromones on paths but also under perturbations. If a new path with a larger eigenvalue will emerge, or a previous food source disappears, the pheromone concentration will adapt and align according to the new conditions.

Using induction on the lines of Sanger's rule (Sanger 1989) it can be shown that if the first $j - 1$ columns of C converge to the first $j - 1$ eigenvectors (or the $j - 1$ most optimal solutions), then the j column will also converge to the j-th eigenvector.

Also, it turns out that in the limit $j = N$, the asymptotic pheromone concentration on the j-th path is proportional to

$$\frac{1}{l_j} \frac{\sum_{p=1}^{N} l_p}{\sum_{p=1}^{N} \lambda_p} , \qquad (13)$$

where λ_p represents the p-th eigenvalue of D.

In the next section, we validate the convergence properties of our algorithm under different conditions that usually inhibit the convergence of ant algorithms to the optimal solution.

Experimental Results

Validation in presence of large bias on sub-optimal paths

The presence of initial bias on suboptimal paths is a critical reason for inability of artificial ant algorithms to converge to the optimal path. To make it difficult for the algorithm to reach the optimal solutions, we initialized the pheromone concentration on each path to be proportional to its length. This implies that initially ants were purposely misguided towards longer paths, thus making it difficult for them to discover and maintain optimal pheromone magnitude on all the paths. Another goal of this experiment was also to validate whether ants are able to optimally regulate the amount of pheromone on all available paths and hence lead to an optimal distribution of ants on the available paths. To verify this we chose $M = N$.

We take a network of 10 alternative paths with lengths increasing from 1 to 10. In each time step, only one ant took a decision. Pheromone trail on each path was tracked and updated according to equation (2) only during the return journey of ants.

Figure 1 shows the change in the pheromone distribution on all the paths during the simulation. Each path is indexed by its length, thus path L_4 was a path of length 4. As seen in the figure, path L_{10}, the path of length 10, was initially given the highest amount of pheromone, of magnitude 1. The shortest path, path L_1 was initially given the least amount of pheromone of 0.1 units. As seen in the figure, ants quickly recover from the biased situation and the pheromone concentrations on the short paths rise while that on the long paths decrease. It is also observed that at steady the concentration of pheromone on each path is in decreasing order of the lengths of the paths. It is also related to the eigenvalue of each path, which is related to the path length for the algorithm.

For example, the set of lengths $l = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10]$, the set of eigenvalues of the corresponding D matrix with $f(l_i) = 1/l_i$, are [1.00000 0.50000 0.33333 0.25000 0.20000 0.16667 0.14286 0.12500 0.11111 0.10000]. The sum of these eigenvalues is 2.9290 and the sum of lengths, $\sum l = 55$. According to equation (13), we expect the asymptotic distribution of pheromone on paths to be proportional to [18.7780 9.3890 6.2593 4.6945 3.7556 3.1297 2.6826 2.3472 2.0864 1.8778]. As seen in the figure, the asymptotic pheromone distribution for all non-optimal paths is not zero, rather it is proportional to the respective eigenvalues.

Thus, it can be summarized that the algorithm leads to the selection of the M best solutions and regulates the quantity of pheromone on each solution such that its magnitude is proportional to the quality of each solution, even when large initial biases are present on sub-optimal paths.

Validation under congestion control and dynamism in path lengths

We elucidate the embedding of an optimization problem by application to dynamic routing. Suppose a network exists, where packets have to be routed from a source to a destination point and several alternate routes or solutions exist that can be used for routing the packets. The goal is to choose the path with the minimum travel time to route the packets. The travel time on each of these routes is governed by several factors like physical length of the network, the speed of the intermediate hubs, the traffic on the paths, any built up of queues etc. In dynamic routing, new paths may arrive and old paths may leave (hub going down), lengths of the paths may vary drastically because of built up of queues etc.

The traversal of an ant on a path is analogous to the sending of a packet on the chosen link. The return of the ant is similar to the receipt of an acknowledgement. We determine the cost c_i of a link by noting the difference in time between
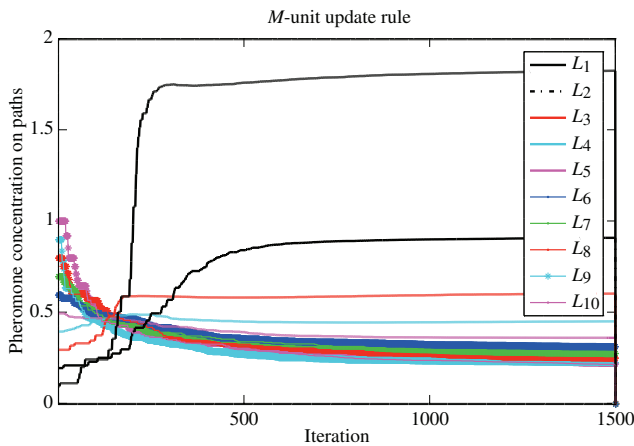
Figure 1: Change in the pheromone distribution on 10 paths of lengths ranging from 1 to 10, and initial biases inversely proportional to their lengths, using the M-unit EigenAnt algorithm
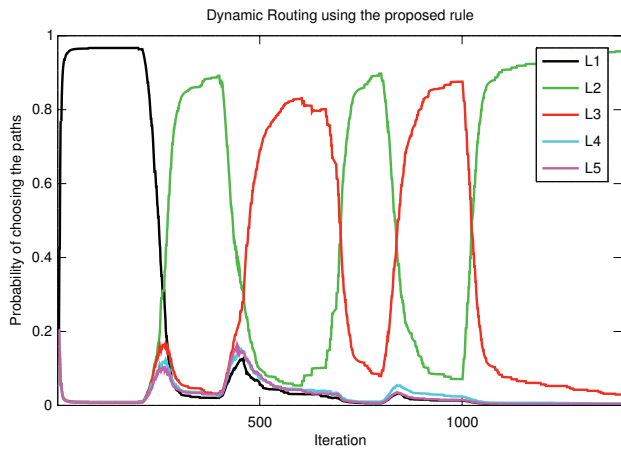

Figure 2: For the problem of dynamically routing packets over a network, this plot shows the change in choosing probability for a random run. The dynamism is the decision making, going down of links and change of lengths of the links. The update rule used is R^1 with $f(l_i) = 1/l_i^3$, $\alpha = 1$, $\beta = 1$. Initially, link L1 was the shortest but it went down. Thereafter, links L2 was periodically introduced heavy traffic, where the algorithm started rerouting most of the traffic to the next best path, path L3 and the congestion on L2 cleared. As seen in the plot, the pheromone concentration reflects the fact that the update rule was able to adapt with the situation.

sending a packet, and receiving its acknowledgement. No additional packets need to be added to the network.

For illustration, we assume that there are 5 distinct paths. Ants choose paths probabilistically according to their pheromone concentrations and update the concentration only on the links that they choose during their backward journey. Figure 2 shows the result of this experiment. We introduced some dynamism after few intervals. Initially all paths were initialized with the same amount of pheromone and path L1 was the shortest path. Therefore, as seen in the figure the concentration quickly builds up on path 1. After a trail had established on it, it suddenly went down at iteration 50. This means that its cost became a very large number. Ants quickly adapted to the situation and started using path 2, the new optimal path. Next, we introduced a large amount of traffic on path 2, such that it could not handle that much traffic and packets started experiencing delay on it. In this situation, path L3 starts getting more reinforcement to adapt with the congestion on path L2. This reduced the traffic on path L2, because of which, at iteration 600, it again became the optimal path with cost = 2. We introduced similar traffic cycles to observe the continual adaptation of the system.

Discussion

In this section, we discuss the individual roles of positive and repellent pheromones. Consider for example two paths of lengths 1 and 2 units with initial pheromone 1 and 10 units on them respectively. Now consider the use of only positive reinforcement. In this case additional pheromone dropped on the longer and shorter path is proportional to $\beta.f(2).(10/11)$ and $\beta.f(1).(1/11)$ respectively. Taking $f() = 1/l_i$ in equation 1, we get $\beta.(5/11)$ on the longer path and $\beta/11$ on the shorter path. Since the initial bias of 10 units on the longer path is more than that on the shorter, more ants will initially choose the longer path. Moreover, each ant will drop a greater amount of pheromone, that is, $\beta.(5/11)$ on the longer path in contrast to $\beta/11$ being dropped on the shorter. This reinforces the longer path which was already "rich" in pheromone "richer" and hence an eventual convergence to the longer path. In this case, the possibility of the shorter path emerging as a winner is probable only if the initial concentration on longer path was up to 2 units. Similarly, in general for a chosen f, there exists an upper limit on amount of initial pheromone that can be reversed by positive reinforcement only (Shah et al. 2008; 2010; Dorigo and Gambardella 1997; Bonabeau, Dorigo, and Theraulaz 1999). The amount of repellent pheromone dropped on the chosen path is proportional to the total pheromone experienced on that path. Repellent pheromones are responsible for eliminating the initial pheromone bias on paths and positive reinforcement assigns more pheromone to better paths.

The parameters α and β weigh the positive and negative feedback terms. It is necessary to keep the value of α lesser than β to reach steady state and build consensus about good paths. Also, if the initial pheromone concentrations on suboptimal paths are very high then α should not be very low compared to β because enough pheromone may not be removed from the sub-optimal paths quickly enough. In other

cases, the ratio β/α simply acts as a scaling factor in the steady state concentrations (cf. equation 6).

Conclusion

Ants are known to use indirect communication via different types of pheromones and follow simple rules. They use only distributed interactions and local information, and distributed decision making. Despite the simplicity, this coordination leads to self-organization and emergence of trails that avoid building of traffic jams, avoid congestion on optimal paths, and use alternative sub-optimal paths and lane formation to obtain a higher global efficiency. For these reasons, they provide valuable insight to self-organization studies that relate simple microscopic interactions with complex emergent macroscopic behavior. In this paper, we shed light on how powerful congestion control based on local interactions may be obtained. We showed how ants can use repellent pheromones and incorporate the effect of crowding to avoid traffic congestion on the optimal path.

Based on these interactions, we proposed an ant algorithm, that leads to the selection of the M shortest paths. To the best of our knowledge, the M-unit EigenAnt algorithm is the first algorithm that explicitly ensures this selection. Infact, this algorithm is in contrast with most ant algorithms that aim to discover just a single best path. Our algorithm should be beneficial in applications that involve any form of congestion control or routing. The algorithm can be also be used to solve a seemingly different problem of foraging from multiple food sources (that is, the optimal simultaneous usage of available resources). If the measure of quality (which is imbibed in the cost matrix itself) incorporates the information of goodness of different resources, then the algorithm is directly applicable.

The distribution of number of ants selecting each of these paths was shown to be optimal in the sense that it is proportional to its shortness (or any measure of quality). We also provided convergence analysis for the steady state distribution of pheromone on all the paths. The steady state distribution was shown to align with the eigenvectors of the cost matrix, and thus related it to its measure of quality.

We also provided perturbation analysis to show that this property ensues even when the food is moved or path lengths change during foraging. If a new better path emerges, then the colony adjusts the proportion of trails according to the new path lengths. We showed that this behavior is robust in the presence of fluctuations and quickly reflects the change in the M optimal solutions. This makes it suitable for not only distributed applications but also dynamic ones as well. Finally, we supported our claims by simulation results that confirm convergence to the optimal solution under different initial biases, dynamism in lengths of paths, and discovery of new paths.

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