Figures for Thought: Temporal Reasoning with Pictures

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Abstract

This paper describes the application of pictorial representations for reasoning about Allen's algebra. We examine four questions:

- Characterizing useful subclasses of relations, such as convex, pointizable, and ORD-Horn relations.
- Elucidating the basic properties of ORD-Horn relations which make them tractable.
- Understanding the nature of relations which are not ORD-Horn.
- Proving the fact that any relation which is not ORD-Horn generates one of four specific relations (corner relations).

We show how using pictorial representations for Allen's relations solves — or helps in solving — those questions.

Similar pictorial techniques can be of value in related fields. The study of concrete cases where diagrams are effectively used in reasoning should result in a deeper understanding of the general domain in Artificial Intelligence concerned with reasoning using pictures and diagrams.

Introduction

This paper is concerned with using diagrams for reasoning. In our view, much insight on how to use picture-based reasoning can be gained by analyzing the way diagrammatic reasoning is *effectively* used by humans in existing fields of research. This study complements the more familiar topic of common-sense reasoning with pictures and diagrams: here, expert knowledge, rather than common-sense knowledge, is involved.

In (Lig97), Ligozat describes the uses of diagrams in a particular branch of mathematics — namely, homological algebra, and more generally category theory. He shows how typically, reasoning with diagrams (in the case he considers, geometrical structures of arrows) involves four steps: representation, construction, inspection and interpretation.

This paper describes analogous processes in the use of graphical representations for reasoning about Allen's relations. Although the basic material presented here has been exposed elsewhere under different forms, it is considered here in a different way: as a case in point of a situation where graphical considerations have an important import on how fairly abstract or arcane notions and results can get a "cognitive conspicuousness" by choosing a representation which sets outs their properties in an illuminating way. It also shows how those representations suggest new results, and how they give hints and suggest strategies for deriving proofs of those results.

Temporal reasoning

Allen's relations

Allen's temporal calculus ((All83)) is concerned with qualitative relations between two intervals in time, as shown in Fig. 1. There are 13 **atomic relations**, denoted by p (precedes), m (meets), o (overlaps), s (starts), d (during), their converses $p^\smile$, $m^\smile$, $o^\smile$, $s^\smile$, $d^\smile$, and eq (equals).

By definition, a **relation** is any subset of the set of atomic relations (usually, no distinction will be made between one-element subsets and the element itself — hence atomic relations are particular cases of relations). Consequently, there are 2^{13} relations, including the void relation $\emptyset$ and the universal relation $\mathbf{1}$ which includes all atoms. The set of relations, equipped with a suitable structure, is called Allen's algebra.

The basic operations in Allen's algebra are intersection, conversion, and composition.

Subclasses in Allen's algebra

Each atomic relation is characterized by a (conjunctive) condition involving the endpoints of the two intervals considered. For instance, for two intervals $I = (I_-, I_+)$ and $J = (J_-, J_+)$, I *overlaps* J if and only if $(I_+ > J_-) \cap (I_+ < J_-)$. More generally, each relation is characterized by a Boolean combination of literals of the form $(I_\alpha \gamma J_\beta)$, where $\alpha, \beta \in \{-,+\}$ and γ is one of $<, >, =, \neq, \leq, \geq$.

Convex relations are those relations which can be expressed by conjunctive formulas, without using $\neq$ ((Nok88; vBC90; Lig91; LM93)).

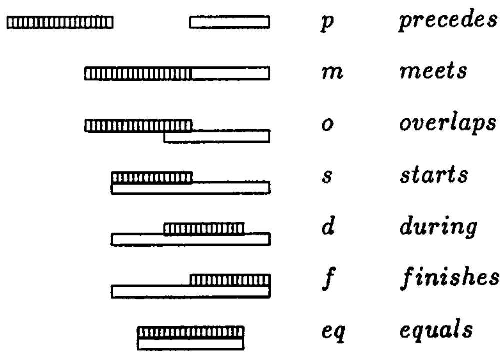

Figure 1: Allen's relations between two intervals

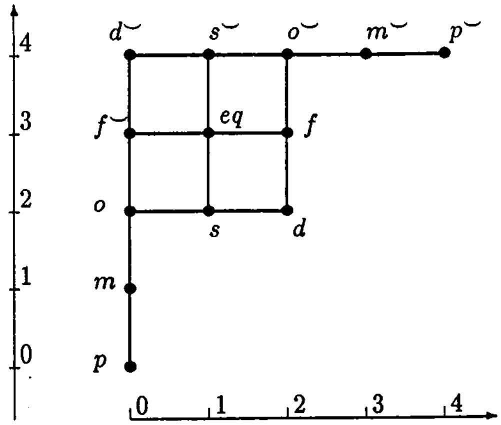

Figure 2: Atomic relations in Allen's algebra

Pointizable relations are those relations which can be expressed by conjunctive formulas ((LM89; vB90; LM93; Lig94a)). .

ORD-Horn relations are those relations which can be expressed by formulas which are Horn clauses, for a suitable definition of positive and negative literals ((NB94; Lig94b)). They include pointizable relations.

Binary constraint networks

Allen's calculus is usually considered in the context of binary constraint networks:

A *binary constraint network* N on Allen's algebra is defined as:

1. A set of nodes $(X_i), 1 \leq i \leq m$.
2. For each pair $(i,j), 1 \leq i,j \leq m$ a relation $\alpha_{i,j}$.

We assume that $\alpha_{j,i}$ is the converse of $\alpha_{i,j}$, and that $\alpha_{i,i}$ is equality, for all i.

A network is *path-consistent* if each label $\alpha_{i,j}$ contains at least an atomic relation, and $(\alpha_{i,j} \circ \alpha_{j,k}) \supseteq \alpha_{i,k}$ for all i, j, k, $1 \leq i,j,k \leq m$.

An *instantiation* of a variable X_i is a pair of real numbers (a_i, b_i), with $a_i < b_i$. If instantiations are given for X_i and X_j, there is exactly one atomic relation r such that the interval (a_i, b_i) is in relation r with respect to (a_j, b_j). We say that the two instantiations of X_i, X_j instantiate r.

A network is *globally consistent* if instantiations can be found for each X_i such that for each pair (i,j) an element in $\alpha_{i,j}$ is instantiated on the edge (i,j). Such a global instantiation is also called a *feasible scenario* for the network. Hence global consistency means that such a feasible scenario exists.

Facts

A central problem in temporal calculus is to determine whether a given network is globally consistent. If it is, one may be interested in determining one (or all) feasible scenarios. A correlated problem is computing the minimal network. The basic facts are the following:

1. Determining consistency in the full calculus is NP hard ((VK86; VKvB89)).
2. Pointizable relations are tractable ((VK86; VKvB89; vB90; LM93)).
3. For convex relations, consistency and the minimal network can be computed with path consistency (ViKa89,vBee90b).
4. For ORD-Horn relations, a path consistent ORD-Horn network is consistent ((NB94; Lig96b)).

Four questions about temporal reasoning

1. How can it be tested that a given relation is convex? pointizable? ORD-Horn?
2. Why are ORD-Horn relations tractable?
3. What do non ORD-Horn relations look like?
4. Why are non ORD-Horn relations not tractable?

This paper shows how full answers or partial, but illuminating answers can be given *pictorially.*

A pictorial excursion through Allen's calculus

The lattice representation

Allen's relations can be represented as in Fig. 2. Each atomic relation is represented as a pair of integers. For example, relation p (precedes) is $(0,0)$, relation m (meets) is $(0,1)$.

Because atomic relations are pairs of integers, there is a partial order on the set of atomic relations, which is given by looking at the order on the co-ordinates: $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. For instance, $p \leq m$. Obviously, p is the smallest element and $p^{\smile}$ the greatest element in this order. In that way, the set of atomic relations is a distributive lattice.

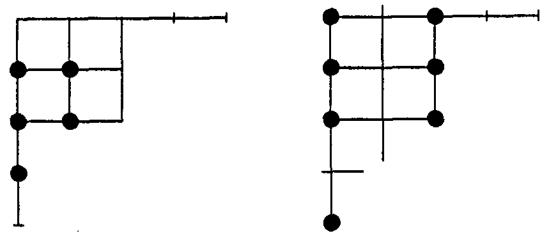

Figure 3: A convex relation (left) and a pointizable, non convex relation (right)

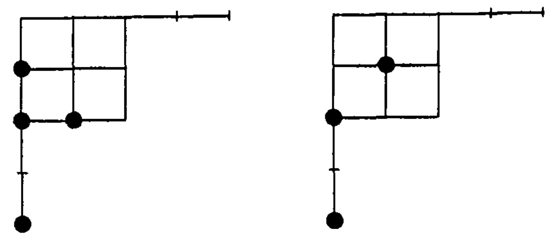

Figure 4: Two ORD-Horn deduced from $[p, eq]$

A pictorial description of the subclasses

We can now answer the first question: How can one test that a given relation is convex? pointizable? ORD-Horn?

Convex relations Choose a pair of atomic relations x, y such that $x \leq y$. Then the set of atomic relations $\{z | x \leq z \leq y\}$ is a relation. It is called the interval defined by x and y, and denoted by $[x, y]$.

Proposition 1 *A relation is* convex *if and only if it is an interval in the lattice. .*

For instance, the relation in the left part of Fig 3 is convex; in fact, it is $[m, eq]$.

Pointizable relations Consider Fig. 2 again. Let us talk about odd verticals for the lines defined by $x_1 = 1$ and $x_1 = 3$. In the same way, odd horizontals are defined by $x_2 = 1$ and $x_2 = 3$. Given a convex relation α, we say we get an odd cut-out of α if we remove from α all the atomic relations lying in some odd verticals and horizontals. For instance, starting from the convex relation $[p, o^{\smile}]$, we can remove all atoms lying in the vertical $x_1 = 1$ and on the horizontal $x_2 = 1$. What we get is the relation $\{p, o, d, f, f^{\smile}, d^{\smile}, o^{\smile}\}$ (see the right part of Fig. 3).

Proposition 2 *A relation is* pointizable *if and only if it is an odd cut-out of a convex relation ((Lig94a)).*

This geometric characterization gives a very simple test for deciding whether a given relation is pointizable. It can also be used to compute the total number of relations without explicitly exhibiting them. Exhaustive enumerations appear in (LM89; vBC90).

ORD-Horn relations We first introduce **stable relations**. Notice that some atomic relations are more stable than others, in the sense that small perturbations of the intervening intervals do not affect them. For instance, p is such a relation: if I precedes J, moving slightly the endpoints will not result in a new atomic relation, provided that the move is small enough. By contrast, m is not stable: any change in the position of I's ending or of J's beginning will result in a new atomic relation (p or o for a small change). Obviously, stable relations are those which do not involve equality between endpoints. Hence, there are six atomic relations of that type, those with two even coordinates: p, o, d and their converses $p^{\smile}$, $o^{\smile}$, $d^{\smile}$.

Let us talk about **unstable** relations for the remaining seven atomic relations.

Proposition 3 *A relation is* ORD-Horn *if and only if it can be obtained from a convex relation by removing only unstable atomic relations ((Lig94b)).*

For instance, consider the convex relation $[p, eq]$. This convex relation contains four unstable atoms, namely m, s, $f^{\smile}$, and eq. Hence it gives rise to $2^4 = 16$ ORD-Horn relations. Two of them are represented in Fig. 4: $\{p, o, s, f^{\smile}\}$ and $\{p, o, eq\}$.

This result gives a very convenient criterion for testing whether a relation is ORD-Horn. It also provides a simple way of computing the total number of ORD-Horn relations. For instance, $[p, eq]$ contains four unstable atoms. Hence it gives rise to $2^4 = 16$ ORD-Horn relations.

Why are ORD-Horn relations tractable?

In (Lig96b), the tractability of ORD-Horn relations is shown by proving that, for any path-consistent ORD-Horn network, a consistent instantiation can be constructed in a backtrack-free way. The two properties which allow the construction to work are:

- Any finite set of intervals in the real line which are pairwise intersecting has a non-empty intersection.
- ORD-Horn relations are "almost" determined by their projections.

The first result is elementary. To give a precise sense to the second result, we introduce a new representation.

The half plane representation

A given interval J is determined by a pair of reals, $J = [a, b]$, with $a < b$, intervals can be considered as points in the plane. If J is fixed, the condition for I to be in one of the atomic relations defines regions in the half plane defined by $X < Y$, as shown in Fig. 5.

Denote by p_X and p_Y the two projections on the X and Y axis respectively. Let H be the upper half plane above the first bisector $X = Y$. A convex relation is

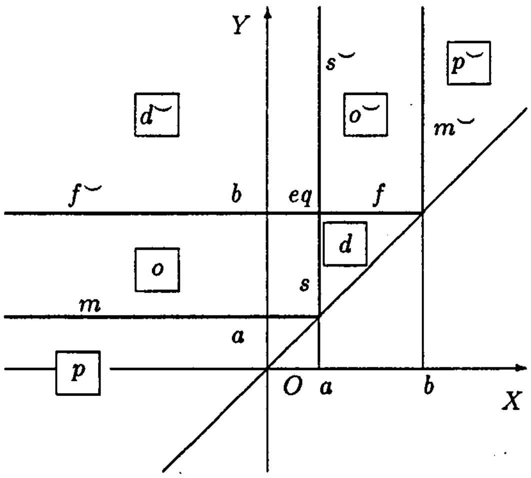

Figure 5: Regions associated to atomic relations in Allen's algebra

convex in the usual sense. Moreover, it is determined by its projections, in the following sense (we denote by α the region in the plane associated to α):

Proposition 4 *A relation α is convex if and only if $pr_X(\alpha)$ and $pr_Y(\alpha)$ are intervals of the real line and $\alpha = H \cap (pr_X(\alpha) \times pr_Y(\alpha)$.*

A consequence for ORD-Horn relations is the following:

Proposition 5 *Proposition If a relation α is ORD-Horn, then $\alpha = H \cap (pr_X(\alpha) \times pr_Y(\alpha)) \setminus \alpha$ is a union of atoms of dimension smaller than the dimension of α.*

A strategy for building scenarios

In (Lig96b), the basic strategy consists in always choosing on each arc of the network an atom which is as stable as possible. In terms of regions, this means that the corresponding region is of maximal dimension. The question is: Why does the strategy work?

The answer is basically because this insures that the interval instantiations belong to the core regions of the relations, rather than to their lower dimensional layers. Hence, although ORD-Horn relations may miss some of those thinner layers, the strategy ensures that each new choice is made in the "safe" part of the relation, where projecting and pulling back are safe.

What do non ORD-Horn relations look like?

We discussed how a suitable representation of ORD-Horn relations was a key to recognizing them and using their closeness to convex relations. We now turn to an illustration of the negative question: what do non ORD-Horn relations look like?

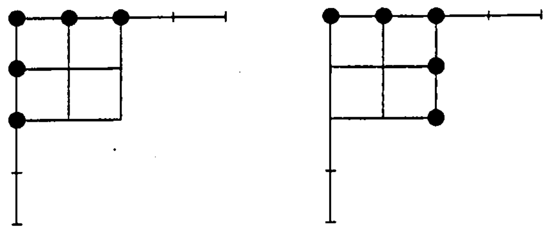

Figure 6: Upper corner relations

Non ORD-Horn implies missing stable relations

The answer is contained in a striking result, which is already almost obvious from the preceding discussion. For any relation α, define its *convex closure* $C(\alpha)$ as the smallest convex relation containing it. Now we have:

Proposition 6 *If α is* not *ORD-Horn, then $C(\alpha) \setminus \alpha$ contains a stable atomic relation.*

In other words, taking the convex closure introduces big relations (represented by two dimensional regions in the plane). In fact, some easy reflection shows that, if p was not in α to begin with, it could not possibly belong to $C(\alpha)$. The same is true for $p^{\smile}$. Hence we get a refinement of the preceding result:

Proposition 7 *If α is not ORD-Horn, then $C(\alpha) \setminus \alpha$ contains one at least of the relations o, d, $o^{\smile}$ and $d^{\smile}$.*

Why are non ORD-Horn relations not tractable?

Corner relations

In the same paper (NB94), Nebel and Bürckert also show that the subclass of ORD-Horn relations is maximal tractable among subclasses containing all atomic relations and which are stable by conversion, intersection, and composition. A key result in their proof is the fact that any such subclass which contains a non ORD-Horn relation must also contain one of four specific relations called "corner" relations by Ligozat (Lig96a):

Definition 1 *The four relations $\{o, f^{\smile}, d^{\smile}, s^{\smile}, o^{\smile}\}$, $\{d^{\smile}, s^{\smile}, o^{\smile}, f, d\}$, $\{d^{\smile}, f^{\smile}, o, s, d\}$, and $\{o, s, d, f, o^{\smile}\}$ are called corner relations (resp. upper-left, upper-right, lower-left, lower-right) (See Fig. 6, 7).*

Corner relations are not ORD-Horn relations: taking the convex closure of any one of them introduces the fourth "missing corner", which is one of the four stable relations o, d, $o^{\smile}$, $d^{\smile}$. In some sense, they are minimally so, since they lack only one of the necessary four relations.

Cornering untractability: The way of all non ORD-Horn

We will indicate briefly how using pictorial representations allows to prove the basic fact used by Nebel and

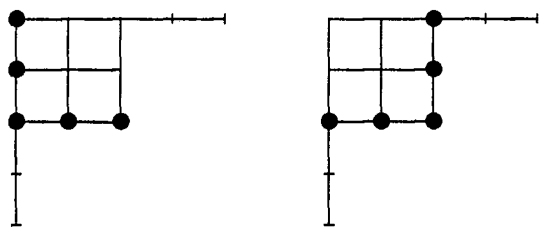

Figure 7: Lower corner relations

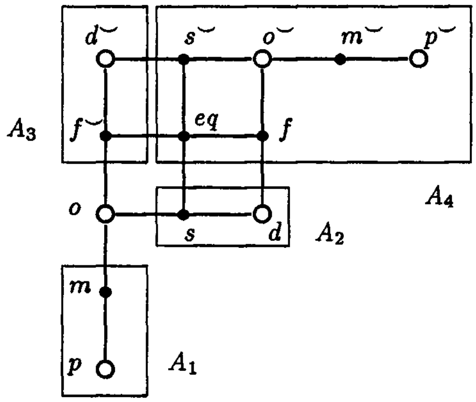

Figure 8: A partition of atomic relations other than o

Bürckert. A full proof appears in (Lig96a). Here we only explain how geometric arguments are involved in the general proof.

Consider a non ORD-Horn relation α. Because α is not ORD-Horn, $C(\alpha) \setminus \alpha$ contains at least one of the four relations o, d, $o^\smile$, $d^\smile$. By using invariance under conversion, we may assume it contains o or d. We sketch a typical step of the proof in the case where it is o (the case where it is d is simpler and uses an analogous argument).

We first partition the set of atomic relations other than o into four distinct subsets A_1, A_2, A_3, A_4 (Fig. 8).

By inspection of Fig. 8 it is fairly clear that, in order for $C(\alpha)$ to contain o, at least one of the following conditions has to hold:

- α does not contain any atom in A_1, and contains atoms in A_2 and in A_3.
- α contains atoms in A_1 and in at least one of the sets A_2, A_3 and A_4.

This is because, else, the convex closure of α could not possibly include o: for instance, if α is contained in $A_2 \cup A_4$, the same is true of $C(\alpha)$.

Now the key to the result lies in the observation of the facts about composition represented in figures 9, 10, 11.

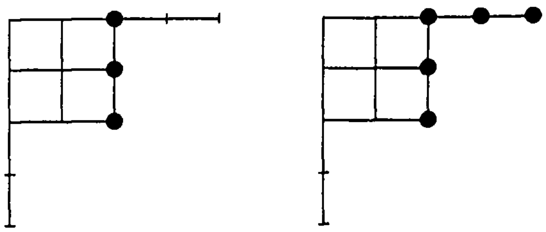

Figure 9: Right composition by $o^\smile$ of elements of A_2: s (left) and d (right)

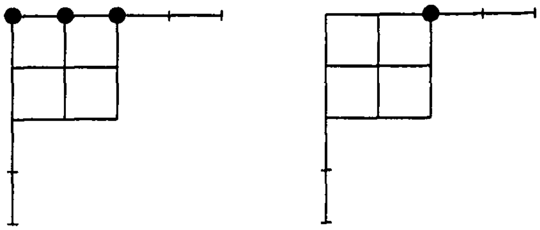

Figure 10: Right composition by $o^\smile$ of elements of A_3: $f^\smile$ and $d^\smile$ (left); right composition by $o^\smile$ of eq and $s^\smile$ (right)

Let us consider the first possibility: α has atoms in both A_2 and A_3. Then, since $o^\smile$ and α belong to the subclass under consideration, their composition also belongs to the subclass. Also, $[o, o^\smile]$ is in the subclass, because it is the composition of two atoms. Hence we can reason on the intersections with $[o, o^\smile]$, i.e. the parts of the relations contained in the central square $[o, o^\smile]$ of the lattice. Now consider figures 9 and 10. Examining them shows that right composition of $o^\smile$ with any element of A_2 implies the presence of $[d, o^\smile]$ in the result, while right composition of $o^\smile$ with any element of A_3 implies the presence of $[d^\smile, o^\smile]$. Finally, Fig. 11 show that elements in A_4 can at most contribute $o^\smile$ to the result. Hence, in all cases, the right upper corner relation belongs to the subclass.

Similar arguments work in the other cases.

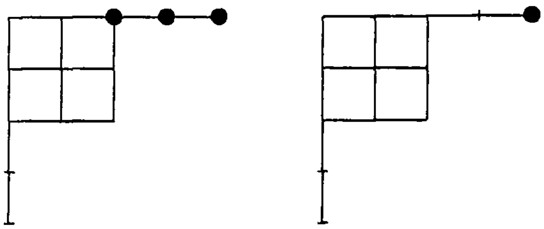

Figure 11: Right composition by $o^\smile$ of eq and $s^\smile$ (left); right composition by $o^\smile$ of f and $o^\smile$ (right)

Conclusions

We showed how four central questions in Allen's calculus can be represented in pictorial terms, and how suitable representations in terms of pictures assist in solving them. For the first question, characterizing important classes of relations, the solution can be read off right away by examination. In the second question, understanding why ORD-Horn relations are tractable was shown to be connected to the geometric properties of a planar representation. The third question, again, was given a simple structural answer. Finally, in the fourth case, we showed how a fundamental result about the algebraic properties of Allen's algebra can also be approached and proved by graphical considerations.

This study suggests further work in two directions:

- Similar considerations should be of interest in related fields, e.g. in dealing either with other models of time and in the field of qualitative spatial reasoning ((Her94)).
- Examining in detail what makes some representations useful for humans is a key component in developing powerful paradigms for diagrammatic reasoning.

References

J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11), 1983.

D. Hernández. Qualitative representation of spatial knowledge. 1994.

Gérard Ligozat. On generalized interval calculi. In *Proceedings of AAAI-91*, pages 234–240, 1991.

Gérard Ligozat. Temporal reasoning made simpler. In *Proceedings of IEA/AIE 94*, pages 123–130, 1994.

Gérard Ligozat. Tractable relations in temporal reasoning: pre-convex relations. In *Proceedings of the ECAI-94 Workshop on Spatial and Temporal Reasoning*, pages 99–108, Amsterdam, 1994.

G. Ligozat. "corner" relations in allen's algebra. In *Working Notes of the AAAI-96 Workshop on Spatial and Temporal Reasoning*, pages 61–65, 1996.

Gérard Ligozat. A new proof of tractability for ord-horn relations. In *Proceedings of AAAI-96*, pages 395–401, 1996.

G. Ligozat. Reasoning with diagrams: the semantics of arrows. 1997. Unpublished.

P. B. Ladkin and R. D. Maddux. On binary constraint networks. 1989.

P. B. Ladkin and R. D. Maddux. On binary constraint problems. *Journal of the ACM*, 41(3):435–469, 1993.

Bernhard Nebel and Hans-Jürgen Bürckert. Reasoning about temporal relations: a maximal tractable subclass of Allen's interval algebra. In *Proceedings of AAAI-94*, pages 356–361, AAAI Press / The MIT Press, Seattle,WA, 1994.

K. Nökel. Convex relations between time intervals. 1988.

Peter van Beek. Reasoning about qualitative temporal information. In *Proceedings of AAAI-90*, pages 728–734, AAAI Press / The MIT Press, Boston, MA, 1990.

P. van Beek and R. Cohen. Exact and approximate reasoning about temporal relations. *Computational Intelligence*, 6:132–144, 1990.

M. B. Vilain and H. Kautz. Constraint propagation algorithms for temporal reasoning. In *Proceedings of AAAI-86*, pages 377–382, 1986.

M. Vilain, H. A. Kautz, and P. G. van Beek. Constraint propagation algorithms for temporal reasoning: a revised report. Morgan Kaufmann, 1989.