A Logic of Branching Histories with a Shared Linear Time Series

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Abstract

I present an expressive temporal logic intended for applications in KR and ontology construction. The formalism combines a 1st-order logic of time with a modal treatment of *historical necessity*, which is used to model describe alternative possible histories. An axiomatisation is given and proved complete with respect to the intended semantics.

Introduction

If the future is indeterminate, then at any time there are many possible futures. This suggests that one might model time in terms of a spreading tree of possible histories, one being the actual history and the others merely possible alternatives. The tenses and moods of our language can be interpreted as locating and relating actual and possible occurrences upon this branching structure of potential happenings.

This is nothing new. Indeed a fair number of possible semantics and axiom sets for branching time structures have already been given (Segerberg 1970; Burgess 1978; 1979; Thomason 1984; Gurevich & Shelah 1985; Zanardo 1985; Di Maio & Zanardo 1994; von Kutschera 1997). Though these works contain quite deep results on branching time logics I have found the formalisms rather difficult to apply to practical problems of knowledge representation. A more AI friendly formalism can be found in (McDermott 1982) but this lacks fully formal semantics and proof theory. The language proposed in the current paper is "yet another branching time temporal logic." However I believe that the specific logical constructs upon which it is based combine expressivity and naturalness in a way that makes it particularly suitable for KR applications and ontology construction.

My formalism is distinguished from all the others I have seen in that it combines a 1st-order treatment of time with a modal treatment of possible histories. The language includes explicit time variables, and time quantification, a construct $@t[\varphi]$, asserting that φ holds at time t, a special predicate $@(t)$, asserting that 't is the actual time'. It also contains a modal 'historical necessity' operator $\Box$, such that $\Box\varphi$ means that φ holds (at the current time) in all histories that share a common past with the actual history.

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The logic is expressive enough to define a wide range of propositional operators relating to tense and possibility. It forms a significant fragment of the much more comprehensive logic presented in (Bennett & Galton to appear). Indeed, the main motivation for the present work was to make some progress towards establishing a complete axiomatisation for that larger system. Unlike most other branching time models, all branches share a common time series. This means we can describe orderings between events in different possible histories. For example, we can represent temporal counterfactuals such as "If I had passed that exam I would now have been president".

Another feature of the language which is attractive for AI applications is its modularity. Unlike many 1st-order temporal languages, time arguments are not added to predicates but instead are attached to propositions using the $@t[\varphi]$ (actually this approach is well-known in AI but little examined in the temporal logic literature). Consequently, atemporal formulas constitute a sub-language of the logic and, where time is not relevant, need not complicate reasoning. Similarly, since the branching structure is made up of a bundle of linear models, where alternative histories are not relevant the logic is equivalent to a simpler linear time logic.

I shall present an axiomatisation for a I give an axiom set and prove completeness of this logic.

The Logic $\mathcal{L}$

Before giving the full branching time logic I present the logic $\mathcal{L}$ of a single linear history. I assume a base logic $\mathcal{L}_B$, which does not contain any time variables or temporal operators. $\mathcal{L}_B$ is used to describe the *states* that hold at the time points referred to within the temporal super-language $\mathcal{L}$. For present purposes we can take $\mathcal{L}_B$ to be ordinary classical propositional or 1st-order logic, but in fact $\mathcal{L}_B$ could be any logic meeting the following requirements: it contains Boolean propositional operators $\neg$ and $\wedge$ (from which $\vee$, $\rightarrow$ and $\leftrightarrow$ can of course be defined); it has a well-defined semantics, such that a model $\mathfrak{B}$ for $\mathcal{L}_B$ assigns one of two values (true or false) to each formula φ of $\mathcal{L}_B$— i.e. $[\![\varphi]\!]_{\mathfrak{B}} \in \{\mathbf{T}, \mathbf{F}\}$; $\neg$ and $\wedge$ have their usual truth functional semantics.

Also, in order that axiomatisation for $\mathcal{L}$ given below is complete we need to have a complete axiomatisation of $\mathcal{L}_B$.

Syntax

The vocabulary of $\mathcal{L}$ extends that of $\mathcal{L}_B$ with a denumerable set of time variables $V = \{\ldots, t_i, \ldots\}$ and some additional logical symbols. All formulae of $\mathcal{L}_B$ are also formulae of $\mathcal{L}$. If φ and ψ are formulae of $\mathcal{L}$ and $t, t' \in V$ then: $\neg\psi$, $\varphi \wedge \psi$, $t = t'$, $t \leq t'$, $\forall t[\varphi]$, $@t[\varphi]$, and $@(t)$ are also formulae of $\mathcal{L}$ (and the constructs $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ and $\exists t[\varphi]$ are added by standard definitions).

Within $\mathcal{L}$ it is very simple to define the well-known propositional past and future tense operators:

- $\mathbf{P}\varphi \equiv_{def} \exists tt'[(@(t) \wedge t' < t) \rightarrow @t'[\varphi]]$
- $\mathbf{F}\varphi \equiv_{def} \exists tt'[(@(t) \wedge t < t't) \rightarrow @t'[\varphi]]$

Semantics

The Boolean connectives and equality have their usual meaning. $\leq$ is a temporal ordering relation and $\forall t$ a quantifier over time points. $@t[\varphi]$ means the proposition φ is true at time t. $@(t)$ means that time variable t refers to the current point in time. Formally, $\mathcal{L}$ is interpreted with respect to a *single history model*, which is a structure $\mathfrak{A} = \langle T, \preceq, B, V, \tau, \{\mathfrak{h}\}\rangle$, where:

- $T = \{\ldots, \mathfrak{t}_i, \ldots\}$ is a set of time values,
- $\preceq$ is a total linear reflexive ordering on T,
- $B = \{\ldots, \mathfrak{B}_i, \ldots\}$ is a set of $\mathcal{L}_B$ model structures,
- $V = \{\ldots, t_i, \ldots\}$ is the set of time variables,
- τ is a mapping from V to T, assigning a time point to each time variable,
- $\mathfrak{h}$ is a mapping from T to B, associating each time point with an $\mathcal{L}_B$ model.

Relative to a model $\mathfrak{A} = \langle T, \preceq, B, V, \tau, \{\mathfrak{h}\}\rangle$ and a time point $\mathfrak{t} \in T$, the denotation $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}}$ of each formula in the language is specified recursively by:

S1) $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = [\![\varphi]\!]_{\mathfrak{B}_\mathfrak{t}}$, where $\varphi \in \mathcal{L}_B$ and $\mathfrak{B}_\mathfrak{t} = \mathfrak{h}(\mathfrak{t})$;
S2) $[\![\neg\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{F}$, else $\mathbf{F}$;
S3) $[\![\varphi \wedge \psi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ and $[\![\psi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$, else $\mathbf{F}$;
S4) $[\![t = t']\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $\tau(t) = \tau(t')$, else $\mathbf{F}$;
S5) $[\![t \leq t']\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $\tau(t) \preceq \tau(t')$, else $\mathbf{F}$;
S6) $[\![@t[\varphi]]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = [\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}'}_{\mathfrak{A}}$, where $\mathfrak{t}' = \tau(t)$;
S7) $[\![@(t)]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $\tau(t) = \mathfrak{t}$, else $\mathbf{F}$;
S8) $[\![\forall t[\varphi]]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = \mathbf{T}$ if $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}^{(t \mapsto \mathfrak{t}')}} = \mathbf{T}$ for all $\mathfrak{t}' \in T$, else $\mathbf{F}$.

The $\mathfrak{h}$ parameter in the denotation function $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}}$ is in fact redundant since the models for $\mathcal{L}$ have only a single history. However, including $\mathfrak{h}$ here will enable us later to generalise the semantics to multiple history structures, without having to re-write all the clauses.

The notation $\mathfrak{A}^{(t \mapsto \mathfrak{t})}$ (in **S8**) refers to a model $\mathfrak{A}' = \langle T, \preceq, B, V, \mathfrak{h}, \tau'\rangle$, which is exactly like $\mathfrak{A}$ except that $\tau'(t) = \mathfrak{t}$, whereas $\mathfrak{A}$ may assign some other time value to the variable t.

Proof System

The logic $\mathcal{L}$ obeys the following general inference rules:

R1) $\vdash \varphi$ if φ is a *valid* formula of $\mathcal{L}_B$.
R2) $\vdash \varphi$ if φ is a classical propositional theorem.
R3) If $\vdash \varphi$ and $\vdash (\varphi \rightarrow \psi)$ then $\vdash \psi$.
R4) If $\vdash (\varphi \rightarrow \psi)$ then $\vdash (\varphi \rightarrow \forall t[\psi])$, provided t does not occur free in φ.
R5) If $\vdash \varphi$ then $\vdash @t[\varphi]$.

If we have a complete axiomatisation of $\mathcal{L}_B$, **R1** can be replaced by the proof rules and axioms of $\mathcal{L}_B$. Rules **R2-4** constitute a standard 1st-order proof system but with quantification restricted to time variables. **R5** ensures that every theorem holds at every time point.

Temporal equality and quantification satisfy standard axiom schemata:

A1) $t = t$,
A2) $(t = t' \wedge \varphi) \rightarrow \varphi^{t \Rightarrow t'}$,
where $\varphi^{t \Rightarrow t'}$ is the result of substituting t' for one or more free occurrences of t in φ ,
A3) $\forall t[\varphi] \rightarrow \varphi^{t \Rrightarrow t'}$,
where $\varphi^{t \Rrightarrow t'}$ is the result of substituting t' for *all* free occurrences of t in φ, and t does not occur in φ within the scope of any quantification w.r.t. the variable t'.

The temporal ordering $\leq$ satisfies the usual axioms for a total linear order:

A4) $(t \leq t' \wedge t' \leq t'') \rightarrow t \leq t''$,
A5) $t \leq t' \vee t' \leq t$,
A6) $(t \leq t' \wedge t' \leq t') \leftrightarrow t = t''$.

The construct $@t[\varphi]$ satisfies the axioms:

A7) $(@t[\varphi] \wedge @t[\varphi \rightarrow \psi]) \rightarrow @t[\psi]$
A8) $\neg @t[\varphi \wedge \neg\varphi]$
A9) $@t[\varphi] \vee @t[\neg\varphi]$
A10) $@t[\varphi] \leftrightarrow @t'[@t[\varphi]]$
A11) $t \leq t' \leftrightarrow @t''[(t \leq t')]$
A12) $\forall t[@t'[\varphi]] \rightarrow @t'[\forall t[\varphi]]$

A7 and **A8** ensure that the formulae that hold at each time point are closed under implication and consistent. **A9** ensures that, for every proposition φ, at each time point either φ or $\neg\varphi$ holds. **A12** means that time quantification commutes with the $@t[\ldots]$ construct.

The predicate $@(t)$ satisfies:

A13) $(@(t) \wedge @(t')) \rightarrow t = t'$
A14) $@t[@(t)]$
A15) $\varphi \rightarrow \exists t[@t[\varphi]]$

The Logic $\mathcal{M}$

I now augment the language $\mathcal{L}$ to accommodate multiple branching histories.

Syntax

The rules of formula construction for $\mathcal{M}$ are exactly the same as for $\mathcal{L}$ except we add the additional clause that, if φ is a formula then $\Box\varphi$ is a formula. We also define the dual operator $\Diamond\varphi \equiv_{def} \neg\Box\neg\varphi$.

Semantics

$\mathcal{M}$ is interpreted relative to a *multiple history model*, a structure $\mathfrak{M} = \langle T, \preceq, B, V, \tau, H\rangle$, where:

- $H = \{\ldots, \mathfrak{h}_i, \ldots\}$ is a set of history functions, each of which maps from T into B,
- For each $\mathfrak{h}_i \in H$, $\mathfrak{M}_{\mathfrak{h}_i} = \langle T, \preceq, B, V, \tau, \{\mathfrak{h}_i\}\rangle$ is a single history model.

This definition means that a single history model is just a special case of multiple history model, where H is a singleton.

To specify the semantics of the logical symbols of $\mathcal{M}$ we employ clauses **S1-8** (for $\mathcal{L}$) without modification. In order to state the semantics of $\Box$ clearly, we define a relation which asserts that two histories are the same up to a given point in time:

$$\mathfrak{h} \overset{\mathfrak{t}}{\approx} \mathfrak{h}' \equiv_{def} (\forall\mathfrak{t}' \in T)[\mathfrak{t}' \preceq \mathfrak{t} \rightarrow \mathfrak{h}(\mathfrak{t}') = \mathfrak{h}'(\mathfrak{t}')]$$

Then we add the stipulation:

S9) $[\![\Box\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{T}$ if $[\![\varphi]\!]^{\mathfrak{h}'\mathfrak{t}}_{\mathfrak{M}} = \mathbf{T}$ for all $\mathfrak{h}' \in H$ such that $\mathfrak{h} \overset{\mathfrak{t}}{\approx} \mathfrak{h}'$, else $\mathbf{F}$.

Thus $\Box\varphi$ is true if φ is true on all histories confluent with the actual history up to the current time. In terms of this several other useful operators can be defined. For example:

- 'Φ is inevitable': $\mathbf{I}\varphi \equiv_{def} \Box\mathbf{F}\varphi$
- 'Φ would have been possible': $\mathbf{W}\varphi \equiv_{def} \mathbf{P}\Diamond\mathbf{F}\varphi$

Axioms for $\mathcal{M}$

$\mathcal{M}$ satisfies all the axioms of $\mathcal{L}$ plus some additional axioms constraining the $\Box$ operator. It satisfies the rule of necessitation:

R6) If $\vdash\varphi$ then $\vdash\Box\varphi$;

and the usual schemata for the modal logic $S5$:

A16) $(\Box\varphi \wedge \Box(\varphi\rightarrow\psi)) \rightarrow \Box\psi$
A17) $\Box\varphi \rightarrow \varphi$
A18) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$

Also, if a formula of the base logic $\mathcal{L}_B$ is true in some history $\mathfrak{h}$ at some time $\mathfrak{t}$ then clearly it will be true in all histories that are confluent up to $\mathfrak{t}$. Thus we have the axiom:

A19) $\alpha \rightarrow \Box\alpha$, for all $\alpha \in \mathcal{L}_B$

The actual time and all temporal inequalities are the same for alternative histories:

A20) $@(t) \rightarrow \Box@(t)$
A21) $t \leq t' \rightarrow \Box(t \leq t')$

Finally we have an axiom fixing the interaction between $\Box$ and the $@t[\varphi]$ construct.

A22) $\forall tt'[@t[\Box\varphi] \wedge t \leq t' \rightarrow @t'[\Box@t[\varphi]]]$

This captures the fact that alternative histories diverge only in the direction of the future.

Since $\Box$ is an $S5$ modality one can derive both the Barcan formula and the converse Barcan formula for $\Box$ with respect to time quantification — i.e. we have the following theorem (the Temporal Barcan Formula):[1]

TBF) $\forall t[\Box\varphi] \leftrightarrow \Box\forall t[\varphi]$

Thus, time quantification commutes with both $@t[\ldots]$ and $\Box$; but $\Box$ and $@t[\ldots]$ do not commute, except in those cases specified by **A22**.

Soundness of $\mathcal{L}$ and $\mathcal{M}$

It is routine to verify that all the axioms are satisfied by every single history model.

Completeness of $\mathcal{L}$

I now give a Henkin-style proof of the completeness of $\mathcal{L}$. The theorem is as follows:

Theorem 1 *The axiom system given for $\mathcal{L}$ is complete with respect to the specified semantics (i.e. every formula that is true in every single history model is provable from the axioms).*

Proof: I shall show in lemma 1 that for every formula γ which is consistent with the axioms there is a single history model, such that γ is true at some time point in that model. Now suppose φ is valid (i.e. true at every time in every model), then $\neg\varphi$ cannot be true at any point in any model and thus $\neg\varphi$ must be inconsistent with the axioms. Consequently φ must be provable. ■.

I now prove the required lemma constructively:

Lemma 1 *For every formula γ which is consistent with the axioms we can construct a single history model, such that γ is true at some time point in that model.*

Proof: For every formula γ that is consistent with the axioms, there is a *maximal consistent* set Γ of $\mathcal{L}$ formulae, which includes γ and is also *existentially closed*. In Lemma 2 I shall show that from every maximal consistent, existentially closed set of formulae Γ we can construct a canonical model $\mathfrak{A}_\Gamma$. Finally, in Lemma 4, I show that for every formula $\varphi \in \Gamma$ (thus in particular for γ), there is a time point in $\mathfrak{A}_\Gamma$ at which φ is true (according to the semantics given for $\mathcal{L}$). ■

The first sentence of the proof needs further explanation. The required properties of Γ are defined as follows:

[1] Those who are new to this language may find **TBF** a little confusing. Bear in mind that $\forall t\varphi$ does *not* mean that φ holds at all time points but that φ holds at the actual time, whatever time point is the value of any free t variables within φ. Thus, $\forall t\Box\varphi$ means that for all values of t, $\Box\varphi$ holds (at the actual time) in all histories confluent with the actual history up to the *actual* time; and hence has the same meaning as $\Box\forall t\varphi$.

- A set of formulae S from a language L is *maximal consistent* iff S is closed under entailment and for each formula $\varphi \in L$ either $\varphi \in S$ or $\neg\varphi \in S$ but not both.
- A set of formulae is existentially closed iff, whenever it contains a formula of the form $\exists x[\varphi]$ it also contains a formula $\varphi^{(x \Rrightarrow u)}$, for some variable u, which is free in $\varphi^{(x \Rrightarrow u)}$.

There are standard techniques to construct sets satisfying these properties and including any given consistent formula. See e.g. (Hughes & Cresswell 1968, p.159). ■

Lemma 2 *From any maximal consistent and existentially closed set* Γ *we can construct a (canonical) single history model* $\mathfrak{A}_\Gamma$.

Proof: I first define some building blocks that will be used to construct the model.

For each time variable t_i in the vocabulary of $\mathcal{L}$, its denotation $\mathfrak{t}_i$ will be the set defined by:

$$\mathfrak{t}_i = \{t_j \mid (t_i = t_j) \in \Gamma\}$$

Thus, the value of each time variable t_i is the equivalence class of those time variables which are equal to t_i according to the equality relations in Γ. Axioms **A1** and **A2** ensure that the sets $\mathfrak{t}_i$ are indeed equivalence classes.

For each $\mathfrak{t}_i$ we construct the following set (which we shall later prove to be the set of all formulae true at $\mathfrak{t}_i$ according to $\mathfrak{A}_\Gamma$):

$$\Gamma_{\mathfrak{t}_i} = \{\varphi \mid @t[\varphi] \in \Gamma \text{ for some } t \in \mathfrak{t}_i\}$$

Since Γ is maximal consistent, **R5**, **A7**, **A8**, and **A9** ensure that each $\Gamma_{\mathfrak{t}_i}$ is also maximal consistent. Furthermore we can show that $\Gamma_{\mathfrak{t}_i}$ is existentially closed: suppose $\exists u[\varphi] \in \Gamma_{\mathfrak{t}_i}$ then $@t[\exists u[\varphi]] \in \Gamma$ for some $t \in \mathfrak{t}_i$. But then using **A12** we can derive $\exists u[@t[\varphi]] \in \Gamma$. So, because Γ is existentially complete $@t[\varphi]^{(u \Rrightarrow v)} \in \Gamma$, for some variable v. Consequently we must have $\varphi^{(u \Rrightarrow v)} \in \Gamma_{\mathfrak{t}_i}$.

I also define

$$\Delta_{\mathfrak{t}_i} = \{\varphi \mid \varphi \in \Gamma_{\mathfrak{t}_i} \wedge \varphi \in \mathcal{L}_B\}\ .$$

So $\Delta_{\mathfrak{t}_i}$ is just the set of non-temporal formulae in $\Gamma_{\mathfrak{t}_i}$. Clearly, $\Delta_{\mathfrak{t}_i}$ is maximal with respect to the language of $\mathcal{L}_B$ and is also consistent. Therefore, for each $\Delta_{\mathfrak{t}_i}$ we can construct an $\mathcal{L}_B$ model satisfying all the formulae in $\Delta_{\mathfrak{t}_i}$. This model will be referred to as $\mathfrak{B}_{\mathfrak{t}_i}$.

From Γ we can construct an $\mathcal{L}$ model

$$\mathfrak{A}_\Gamma = \langle T_\Gamma, \preceq_\Gamma, B_\Gamma, V_\Gamma, \tau_\Gamma, \{\mathfrak{h}_\Gamma\}\rangle$$

whose components are specified as follows:

- V_Γ is the set of time variables of the language $\mathcal{L}$.
- The function τ_Γ which assigns values to the time variables is defined by $\tau_\Gamma(t_i) = \mathfrak{t}_i$.
- The set T_Γ of all time points is defined to be the range of τ_Γ.
- The ordering relation $\preceq_\Gamma$ is defined by:

  $$\mathfrak{t}_i \preceq_\Gamma \mathfrak{t}_j \text{ iff } (\forall t \in \mathfrak{t}_i)(\forall t' \in \mathfrak{t}_j)[(t \leq t') \in \Gamma]$$

  Because the time points are quivalence classes of the time variables and Γ is maximal and closed under **A4-6**, it follows that $\preceq_\Gamma$ must be a total linear order.
- We set $\mathfrak{h}_\Gamma(\mathfrak{t}_i) = \mathfrak{B}_{\mathfrak{t}_i}$
- Finally, we let B_Γ be equal to the range of $\mathfrak{h}_\Gamma$. ■

The following equivalence which relates formulae in Γ to formulae in the derived sets $\Gamma_{\mathfrak{t}_i}$ will be used several time (sometimes without reference) in the remainder of the proof:

Lemma 3 *The following conditions are equivalent:*

- $\varphi \in \Gamma_{\mathfrak{t}_i}$,
- $@t[\varphi] \in \Gamma$ *for every variable* $t \in \mathfrak{t}_i$,
- $@t[\varphi] \in \Gamma$ *for some variable* $t \in \mathfrak{t}_i$.

Proof: By definition, $\varphi \in \Gamma_{\mathfrak{t}_i}$ just in case $@t[\varphi] \in \Gamma$ for all $t \in \mathfrak{t}_i$; and moreover $\mathfrak{t}_i$ is by definition non-empty. On the other hand if $\varphi \notin \Gamma_{\mathfrak{t}_i}$ then, because $\Gamma_{\mathfrak{t}_i}$ is maximal consistent, $\neg\varphi \in \Gamma_{\mathfrak{t}_i}$, which means that $@t[\neg\varphi] \in \Gamma$ for all $t \in \mathfrak{t}_i$. From **R5**, **A7** and **A8**, we can derive $@t[\neg\varphi] \to \neg@t[\varphi]$. Thus, because Γ is closed under implication $\neg@t[\varphi] \in \Gamma$; so, since Γ is consistent $@t[\varphi] \notin \Gamma$ for any $t \in \mathfrak{t}_i$.

Because the variable set $\mathfrak{t}_i$ is by definition non-empty, the third condition follows from the second. ■

Lemma 4 *In the cannonical model* $\mathfrak{A}_\Gamma$ *constructed from any maximal consistent, existentially closed formula set* Γ. *For every formula* $\varphi \in \Gamma$ *there is a time point at which* φ *is are true according to the semantics given for* $\mathcal{L}$.

Proof: By **A15** and the maximal consistency of Γ, we know that $\varphi \in \Gamma$ implies $\exists t[@t[\varphi]] \in \Gamma$ and hence, because Γ is existentially closed, $@u[\varphi] \in \Gamma$ for some time u. Thus, by Lemma 3, $\varphi \in \Gamma_{\mathfrak{t}_i}$, where $\mathfrak{t}_i = \tau_\Gamma(u)$ is the denotation of u. Below, in Lemma 5, I shall prove that $\varphi \in \Gamma_{\mathfrak{t}_i}$ just in case $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$. Therefore φ is true at time $\mathfrak{t}_i$ in $\mathfrak{A}_\Gamma$. ■.

Now we come to the final requirement of the proof, which is to demonstrate the following lemma:

Lemma 5 *For each* $\mathfrak{t}_i$ *the set* $\Gamma_{\mathfrak{t}_i}$ *contains exactly those formulae which are true at* $\mathfrak{t}_i$ *according to* $\mathfrak{A}_\Gamma$ *— i.e.* $\varphi \in \Gamma_{\mathfrak{t}_i}$ *if and only if* $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$.

Proof: The proof is achieved by induction on the structure of $\mathcal{L}$ formulae:

Non-Temporal Case: Consider first the case where $\varphi \in \mathcal{L}_B$. If $\varphi \in \Gamma_{\mathfrak{t}_i}$ then $\varphi \in \Delta_{\mathfrak{t}_i}$, which by definition is satisfied by $\mathfrak{B}_{\mathfrak{t}_i}$. Thus, **S1** means that $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = [\![\varphi]\!]_{\mathfrak{B}_{\mathfrak{t}_i}} = \mathbf{T}$. On the other hand, if $\varphi \notin \Gamma_{\mathfrak{t}_i}$ then, because $\Gamma_{\mathfrak{t}_i}$ is maximal, $\neg\varphi \in \Gamma_{\mathfrak{t}_i}$. Now since $\neg$ is a connective of $\mathcal{L}_B$, $\neg\varphi$ is also in $\mathcal{L}_B$, and thus in $\Delta_{\mathfrak{t}_i}$. Therefore $[\![\neg\varphi]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = [\![\neg\varphi]\!]_{\mathfrak{B}_{\mathfrak{t}_i}} = \mathbf{T}$ and consequently $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = [\![\varphi]\!]_{\mathfrak{B}_{\mathfrak{t}_i}} = \mathbf{F}$.

Case of $t = t'$ **and** $t \leq t'$ **Atoms:** If $t = t' \in \Gamma_{\mathfrak{t}_i}$ then by **A6** $t \leq t'$, $t' \leq t \in \Gamma_{\mathfrak{t}_i}$. So for some times u and u' $@u[t \leq t']$, $@u'[t' \leq t] \in \Gamma$ and thus by **A11** $t \leq t'$, $t' \leq t \in \Gamma$. Then using **A6** again we get $t = t' \in \Gamma$. From the definition of τ_Γ and the stipulation **S4** we must then have $[\![t = t']\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$. On the other hand, if $t = t' \notin \Gamma_{\mathfrak{t}_i}$ then $\neg(t = t') \in \Gamma_{\mathfrak{t}_i}$. So because of maximality and **A6** we must have either $\neg(t \leq t') \in \Gamma_{\mathfrak{t}_i}$ or $\neg(t' \leq t) \in \Gamma_{\mathfrak{t}_i}$. Hence, either $\neg(t \leq t') \in \Gamma$ or $\neg(t' \leq t) \in \Gamma$, so because of

A6 we get $\neg(t = t') \in \Gamma$ and consequently $t = t' \notin \Gamma$. Referring once more to the definition of τ_Γ and to **S4** we see that $[\![t = t']\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$.

Suppose $t \leq t' \in \Gamma_{\mathfrak{t}_i}$. This means that $@u[t \leq t'] \in \Gamma$ for some $u \in \mathfrak{t}_i$ and, by **A11**, $t \leq t' \in \Gamma$. From the definition of τ_Γ and $\preceq_\Gamma$ and the stipulation **S5** we must then have $[\![t \leq t']\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$. On the other hand, if $t \leq t' \notin \Gamma_{\mathfrak{t}_i}$, then by maximality $\neg(t \leq t') \in \Gamma_{\mathfrak{t}_i}$ and so, by **A5**, $t' \leq t \in \Gamma_{\mathfrak{t}_i}$. Then, following the same reasoning used for the positive case, we now get (1) $t' \leq t \in \Gamma$. Also, if $\neg(t \leq t') \in \Gamma_{\mathfrak{t}_i}$ then by **A6** $\neg(t = t') \in \Gamma_{\mathfrak{t}_i}$ so $t = t' \notin \Gamma_{\mathfrak{t}_i}$. Thus, as shown in the previous paragraph, we get (2) $\neg(t = t') \in \Gamma$. From (1) and (2) and **A6** we derive $\neg(t \leq t') \in \Gamma$, so $t \leq t' \notin \Gamma$. Then from **S5** we get $[\![t \leq t']\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$.

Case of $@(t)$**:** According to the semantics: $[\![@(t)]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$ iff $\tau_\Gamma(t) = \mathfrak{t}_i$. So we need to show that $@(t) \in \Gamma_{\mathfrak{t}_i}$ iff $\tau_\Gamma(t) = \mathfrak{t}_i$.

Suppose $@(t) \in \Gamma_{\mathfrak{t}_i}$. Then for any $t' \in \mathfrak{t}_i$ we know that $\tau_\Gamma(t') = \mathfrak{t}_i$ and $@t'[@(t)] \in \Gamma$. But then, because of axiom **A14**, we must also have $@t'[@(t')] \in \Gamma$, and hence, $@t'[@(t) \wedge @(t')] \in \Gamma$. Now we can appeal to **A13** to get $@t'[t = t'] \in \Gamma$ and **A11** to get $t = t' \in \Gamma$. Then, from the definition of τ_Γ, we have $\tau_\Gamma(t) = \tau_\Gamma(t') = \mathfrak{t}_i$.

On the other hand, if $@(t) \notin \Gamma_{\mathfrak{t}_i}$, then for all $t' \in \mathfrak{t}_i$ $\neg @t'[@(t)] \in \Gamma$. Now suppose that $\tau_\Gamma(t) = \mathfrak{t}_i$. Because of the definition of τ_Γ, this would mean that $t = t' \in \Gamma$, and from this using **A14** and **A2** we can derive $@t'[@(t)] \in \Gamma$. This is impossible since Γ is maximal consistent. Therefore we must have $\tau_\Gamma(t) \neq \mathfrak{t}_i$.

Since we have proved Lemma 1 for all formulae of $\mathcal{L}_B$ and for the atomic formulae of $\mathcal{L}$, we use this as the base case to prove it by induction for all $\mathcal{L}$ formulae.

Case of the Boolean Connectives: Suppose $\neg\varphi \in \Gamma_{\mathfrak{t}_i}$. By the induction hypothesis we can assume that $[\![\varphi]\!]^{\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$ iff $\varphi \in \Gamma_{\mathfrak{t}_i}$. So, since $\neg\varphi \in \Gamma_{\mathfrak{t}_i}$, consistency ensures we have $\varphi \notin \Gamma_{\mathfrak{t}_i}$, and hence $[\![\varphi]\!]^{\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$. Then, according to **S2**, we have $[\![\neg\varphi]\!]^{\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$. On the other hand if $\neg\varphi \notin \Gamma_{\mathfrak{t}_i}$ maximality ensures that $\varphi \in \Gamma_{\mathfrak{t}_i}$. So then $[\![\varphi]\!]^{\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$ and $[\![\neg\varphi]\!]^{\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$.

The case of conjunctions $(\varphi \wedge \psi)$ is similarly trivial.

Case of $@t[\varphi]$**:** According to **S6**, $[\![@t[\varphi]]\!]^{\mathfrak{t}}_{\mathfrak{A}_\Gamma} = [\![\varphi]\!]^{\tau_\Gamma(t)}_{\mathfrak{A}_\Gamma}$. The induction hypothesis then allows us to assume that $[\![\varphi]\!]^{\tau_\Gamma(t)}_{\mathfrak{A}_\Gamma} = \mathbf{T}$ iff $\varphi \in \Gamma_{\tau_\Gamma(t)}$, so we just need to show that $@t[\varphi] \in \Gamma_{\mathfrak{t}_i}$ iff $\varphi \in \Gamma_{\tau_\Gamma(t)}$. Suppose $@t[\varphi] \in \Gamma_{\mathfrak{t}_i}$. Then $@t'[@t[\varphi]] \in \Gamma$ for each $t' \in \mathfrak{t}_i$. But because of **A10** this means that we also have $@t[\varphi] \in \Gamma$ and hence, from the definitions of $\tau_\Gamma(t)$ and the sets $\Gamma_{\mathfrak{t}_i}$, we must have $\varphi \in \Gamma_{\tau_\Gamma(t)}$. On the other hand, if $@t[\varphi] \notin \Gamma_{\mathfrak{t}_i}$, then, because of maximal consistency, $\neg @t[\varphi] \in \Gamma_{\mathfrak{t}_i}$; and thus, by **A9**, $@t[\neg\varphi] \in \Gamma_{\mathfrak{t}_i}$. This means that $@t'[@t[\neg\varphi]] \in \Gamma$ for each $t' \in \mathfrak{t}_i$; and, because of **A10**, we also have $@t[\neg\varphi] \in \Gamma$. Consequently, $\neg\varphi \in \Gamma_{\tau_\Gamma(t)}$ and so, as required, $\varphi \notin \Gamma_{\tau_\Gamma(t)}$.

Case of $\forall t[\varphi]$**:** Suppose $\forall t[\varphi] \in \Gamma_{\mathfrak{t}_i}$. According to **S8**, $[\![\forall t[\varphi]]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$ just in case $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}(t\mapsto\mathfrak{t}')} = \mathbf{T}$ for all $\mathfrak{t}' \in T$. By the definition of T_Γ, for every $\mathfrak{t}' \in T_\Gamma$ there is an equation $t' = t'$ in Γ such that $\tau_\Gamma(t') = \mathfrak{t}'$; so every element in T_Γ is denoted by some time variable. Consider any time variable t' such that $\tau_\Gamma(u) = \mathfrak{t}'$ and let $\varphi^{(t'\Rightarrow\backslash t)}$ be an alphabetic variant of φ such that any t' occurring in a quantifier or bound by a quantifier within φ is replaced by a new variable that is distinct from t and does not occur anywhere in φ. (The purpose of constructing this variant of φ is to get an equivalent formula for which we can substitute t' in place of t without fear of t' becoming bound by some quantifier in φ.) By standard classical reasoning using **R2-R4** and **A3** one can show that $\varphi^{(t'\Rightarrow\backslash t)} \leftrightarrow \varphi$ and hence (after some routine manipulation of the Boolean connectives we see that) in every model these formulae are assigned the same truth value. It is then easy to see that: $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}(t\mapsto\mathfrak{t}')} = [\![\varphi^{(t'\Rightarrow\backslash t)}]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}(t\mapsto\mathfrak{t}')} = [\![(\varphi^{(t'\Rightarrow\backslash t)})^{(t\Rrightarrow t')}]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}}$

By some further simple classical reasoning we can also show that $\forall t[\varphi^{(t'\Rightarrow\backslash t)}] \leftrightarrow \forall t[\varphi]$. Therefore, since, $\forall t[\varphi] \in \Gamma_{\mathfrak{t}_i}$ we also have $\forall t[\varphi^{(t'\Rightarrow\backslash t)}] \in \Gamma_{\mathfrak{t}_i}$. Then using **A3** we can derive $(\varphi^{(t'\Rightarrow\backslash t)})^{(t\Rrightarrow t')} \in \Gamma_{\mathfrak{t}_i}$ for any time variable t'. (Note that $\varphi^{(t'\Rightarrow\backslash t)}$ must satisfy the proviso on axiom **A3**.)

The induction on formula structure can be run in such a way that, when we need to show Lemma 1 holds for a formula with n quantifiers, we have already proved it for all formulae with $n-1$ quantifiers. Consequently we can assume that Lemma 1 holds for all substitution instances of all alphabetic variants of φ, and hence for any formula $(\varphi^{(t'\Rightarrow\backslash t)})^{(t\Rrightarrow t')}$. Thus, $[\![(\varphi^{(t'\Rightarrow\backslash t)})^{(t\Rrightarrow t')}]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}} = [\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}(t\mapsto\mathfrak{t}')} = \mathbf{T}$, for all $\mathfrak{t}' \in T_\Gamma$; and therefore, as required $[\![\forall t[\varphi]]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{T}$.

Suppose on the other hand $\forall t[\varphi] \notin \Gamma_{\mathfrak{t}_i}$; then $\neg\forall t[\varphi] \in \Gamma_{\mathfrak{t}_i}$ and hence $\exists t[\neg\varphi] \in \Gamma_{\mathfrak{t}_i}$. Thus, because $\Gamma_{\mathfrak{t}_i}$ is existentially closed, there is some time variable u such that $\neg\varphi^{(t\Rrightarrow u)} \in \Gamma_{\mathfrak{t}_i}$. Thus $\varphi^{(t\Rrightarrow u)} \notin \Gamma_{\mathfrak{t}_i}$ and $[\![\varphi^{(t\Rrightarrow u)}]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$. Hence we must also have $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{A}(t\mapsto\mathfrak{t}')} = \mathbf{F}$, for the case where $\mathfrak{t}' = \tau_\Gamma(u)$; and, according to **S8**, this means that $[\![\forall t[\varphi]]\!]^{\mathfrak{h}\mathfrak{t}_i}_{\mathfrak{A}_\Gamma} = \mathbf{F}$.

This case by case analysis completes the proof of Lemma 5. ■

Having proved all the required lemmas, we have now established Theorem 1.

Completeness of $\mathcal{M}$

I again employ a Henkin-style approach to show that γ is consistent with the axioms then there is a multiple history model which satisfies it. Though the following proof is already rather complex, there are still some places where it glosses over the full details. Construction of a completely thorough proof is ongoing. Hopefully some simplification will also be possible.

TME-Sets

In order to model the semantics of the modal operators I construct maximal consistent sets that are not only existentially closed but also contain a stock of certain kinds formulae which will facilitate the derivation of other formulae sets corresponding to alternative histories. I modify (by adding a temporal component) the method of (Hughes & Cresswell 1968, Chapter 9), where it is shown how to construct sets having the so-called 'E'_M-*property*', which enables existential closure to be transported to modal alternatives. For the current proof I introduce the closely related idea of a *TME-set*, a set of formulae containing a certain class of *Temporal Modal Existential Formulae* (henceforth *TME-formulae*).

A *TME-formula with respect to substitution variable* u is any formula in that satisfies the following recursive specification:

- Any formula of the form $@t[\exists x[\varphi]] \rightarrow @t[\varphi^{(x \Rrightarrow u)}]$ is a TME-formula with respect ot u.
- If ξ_u is a TME-formula w.r.t. u and ψ is any formula that does not contain u free, then any formula of the form $@t[\Diamond\,\psi] \rightarrow @t[\Diamond(\psi \wedge \xi)]$ is a TME-formula w.r.t. u.

Lemma 6 *If ξ_u is a TME-formula with respect to u then $\exists u[\xi_u]$ is a theorem of $\mathcal{M}$.*

Proof: We perform induction on the definition of a TME-formula. First the base case. From simple predicate logic and **A7** it follows that $\vdash\ @t[\exists x[\varphi]] \rightarrow @t[\exists u[\varphi^{(x \Rrightarrow u)}]]$. Then using **A12** we get $\vdash\ @t[\exists x[\varphi]] \rightarrow \exists u[@t[\varphi^{(x \Rrightarrow u)}]]$. Since, u does not occur free in the antecedent, this can be rewritten to get $\vdash\ \exists u[@t[\exists x[\varphi]] \rightarrow @t[\varphi^{(x \Rrightarrow u)}]]$.

Now to prove the induction step we show that, if $\vdash\ \exists u[\xi_u]$ and ψ does not contain u free, then $\vdash \exists u[@t[\Diamond\,\psi] \rightarrow @t[\Diamond(\psi \wedge \xi_u)]]$. Since $\vdash\ \exists u[\xi_u]$ it follows from the $S5$ axioms that $\vdash \Diamond\,\psi \rightarrow\ \Diamond(\psi \wedge \exists u[\xi_u])$; and then from **A7** that $\vdash @t[\Diamond\,\psi] \rightarrow @t[\Diamond(\psi \wedge \exists u[\xi_u])]$. Then, using **TBF** and **A12** together with the fact that u does not occur free in ψ, we can move the existential quantifier outwards to obtain $\vdash\ \exists u[@t[\Diamond\,\psi] \rightarrow @t[\Diamond(\psi \wedge \xi_u)]]$. Induction then proves the lemma for all TME-formulae. ■

We can now show that:

Lemma 7 *For any finite consistent set Λ and any TME-formula ξ_u with respect to a variable u that does not occur in any formula in Λ, the set $\Lambda \cup \{\xi_u\}$ is also consistent.*

Proof: This follows from Lemma 6 by standard 1st-order reasoning — see e.g. (Hughes & Cresswell 1968, p. 160). ■

We now define TME-forms and TME-sets as follows:

- A *TME-form* is a maximal set of all TME-formulae that are identical apart from the substitution variable.
- A *TME-set* is any set of formulae that contains at least one member of each TME-form.

Lemma 8 *Any consistent finite formula set can be expanded to a consistent TME-set (and therefore to a maximal consistent TME-set).*

Proof: Start with any finite consistent formula set. Arrange the TME-forms in some sequence and then successively add from each TME-form a TME-formula with respect to some variable that does not already occur in any formula so-far contained in the set (since we have a denumerable number of variables, there is always a new variable available). Lemma 7 ensures consistency of the resulting TME-set. (This consistent TME-set can then be expanded to a maximal consistent TME-set in the usual way). ■

$t\alpha$-Alternatives

Suppose we have a maximal consistent TME-set Γ consisting of all the formulae true at some time on some history $\mathfrak{h}$; and suppose that Γ contains a formula $@t[\Diamond\,\alpha]$. According to the semantics of $\mathcal{M}$ this means there must be some other history $\mathfrak{h}'$, which is confluent with $\mathfrak{h}$ up to a time $\mathfrak{t} = \tau(t)$, and such that α is true at $\mathfrak{t}$ on $\mathfrak{h}'$. Let Γ' be the set of formulae true at $\mathfrak{t}$ on $\mathfrak{h}'$. The semantics of $\Box$ requires that for every formula $@t[\Box\,\varphi_i] \in \Gamma$ we must have $\psi_i \in \Gamma'$. To characterise this relationship we define the following relation:

- Γ' is a $t\alpha$*-alternative* to Γ just in case: both Γ and Γ' are maximal consistent TME-sets, $@t[\Diamond\,\alpha] \in \Gamma$, $\alpha \in \Gamma'$ and for each $@t[\Box\,\varphi_i] \in \Gamma$ we have $\psi_i \in \Gamma'$.

Lemma 9 *Every maximal consistent TME-set containing a formula $@t[\Diamond\,\alpha]$ has a $t\alpha$-alternative.*

Proof: Let Γ be a maximal consistent TME-set containing $@t[\Diamond\,\alpha]$. We will construct a $t\alpha$-alternative Γ' by extending the set $\{\alpha\}$ to a maximal TME-set in the following way. We arrange all the TME-forms in some order and index them as $\Xi_1 \ldots \Xi_n \ldots$. We must pick one TME-formula from each Ξ_1; and we will constrain this choice by reference to the formulae in Γ. Since Γ is a TME-set, it must contain a formula $@t[\Diamond\,\alpha] \rightarrow @t[\Diamond(\alpha \wedge \xi_1)]$ for some $\xi_1 \in \Xi_1$. Moreover, Γ must also contain a formula $@t[\Diamond(\alpha \wedge \xi_1)] \rightarrow @t[\Diamond(\alpha \wedge \xi_1 \wedge \xi_2)]$ for some $\xi_2 \in \Xi_2$; and more generally, given any choices for $\xi_1 \in \Xi_1 \ldots, \xi_n \in \Xi_n$ we can pick $\xi_{n+1} \in \Xi_{n+1}$ such that

$$@t[\Diamond(\alpha \wedge \xi_1 \wedge\ \ldots\ \wedge \xi_n)] \rightarrow @t[\Diamond(\alpha \wedge \xi_1 \wedge\ \ldots\ \wedge \xi_n \wedge \xi_{n+1})]$$

is in Γ. Following this selection procedure we derive an infinite sequence of formulae ξ_n one from each Ξ_n. Hence, the infinite set $\Lambda = \{\alpha, \xi_1, \ldots\}$ is a TME-set. Because Γ is maximal consistent (and hence is closed under modus ponens and contains every instance of the theorem schema $@t[\Diamond(\psi \wedge \psi')] \rightarrow @t[\Diamond(\psi)]$), we see that for any subset $\{\lambda_1, \ldots, \lambda_j\}$ of Λ, the formula $@t[\Diamond(\lambda_1 \wedge\ \ldots\ \wedge \lambda_j)]$ is in Γ.

Now let $\Pi = \{\varphi_i \mid @t[\Box\,\varphi_i] \in \Gamma\}$. We can show that $\Lambda \cup \Pi$ is consistent. For any finite subset $S = \{\lambda_1, \ldots, \lambda_m, \varphi_1, \ldots, \varphi_n\} \subset (\Lambda \cup \Pi)$. we see that Γ must then contain the formulae

$$@t[\Diamond(\alpha, \lambda_1 \wedge\ \ldots\ \wedge \lambda_m)], @t[\Box\,\varphi_1], \ldots @t[\Box\,\varphi_n].$$

But then if S were inconsistent, simple reasoning using **A7-9** and the $S5$ axioms would show that Γ is inconsistent. However, by assumption Γ is consistent and thus S must be consistent. Moreover, since S is an arbitrary finite subset of $\Lambda \cup \Pi$ then $\Lambda \cup \Pi$ must itself be consistent. So finally we can extend $\Lambda \cup \Pi$ to a maximal consistent TME-set Γ', which must be a $t\alpha$-alternative to Γ. ■

Alternative Sequences

In constructing $\mathcal{M}$ models we shall be concerned with formula sets that are derived from an initial set *via* a sequence of $t\alpha$-alternative relationships. Thus we define:

- An *alternative sequence* derived from Γ_0 is a finite sequence

  $$\langle\langle\emptyset, \emptyset, \Gamma_0\rangle, \langle t_1, \alpha_1, \Gamma_1\rangle, \ldots, \langle t_n, \alpha_n, \Gamma_n\rangle\rangle\ ,$$

  where (for $n > 0$) each Γ_n is a $t_n\alpha_n$-alternative to Γ_{n-1} (hence $\alpha_n \in \Gamma_n$ and $@t[\Diamond\,\alpha_i] \in \Gamma_{n-1}$).

The following lemma will be used later for reasoning about the propagation of formulae through a sequence:

Lemma 10 *For any sequence* $\Sigma_i = \langle \ldots, \langle t_{n-1}, \alpha_{n-1}, \Gamma_{n-1}\rangle, \langle t_n, \alpha_n, \Gamma_n\rangle, \ldots\rangle$, $@t_n[\Box\,\varphi] \in \Gamma_{n-1}$ *if and only if* $@t_n[\Box\,\varphi] \in \Gamma_n$.

Proof: From **A14** we have $\vdash\ @t_n[@(t_n)]$ and from **A20**, $@(t_n) \rightarrow \Box\,@(t_n)$, therefore by **A7** we get 1) $\vdash @t_n[\Box\,@(t_n)]$. From the $S5$ axioms and **A7** we get $@t_n[\Box\,\varphi]\ \vdash\ @t_n[\Box\,\Box\,\varphi]$. So, combining this with 1), we have $@t_n[\Box\,\varphi]\ \vdash\ @t_n[\Box\,\Box\,\varphi] \wedge @t_n[\Box\,@(t_n)]$. From this we get $@t_n[\Box\,\varphi]\ \vdash\ @t_n[\Box(\Box\,\varphi \wedge @(t_n))]$. Hence if $@t_n[\Box\,\varphi]\ \in\ \Gamma_{n-1}$ so is $@t_n[\Box(\Box\,\varphi \wedge @(t_n))]$ and, since Γ_n is a $t\alpha$-alternative to Γ_{n-1}, the construction rule for sequences gives us $\Box\,\varphi \wedge @(t_n) \in \Gamma_n$. Using **A15** and **A13**, we then get $\Box\,\varphi \wedge @(t_n) \in \Gamma_n \vdash @t_n[\Box\,\varphi]$. So we must have $@t_n[\Box\,\varphi] \in \Gamma_n$.

On the other hand suppose $@t_n[\Box\,\varphi]\ \notin\ \Gamma_{n-1}$, then $\neg @t_n[\Box\,\varphi]\ \in\ \Gamma_{n-1}$ and from this we get $@t_n[\Diamond\,\neg\varphi]\ \in \Gamma_{n-1}$. So using **A18** and **A7** we must have $@t_n[\Box\,\Diamond\,\neg\varphi] \in \Gamma_{n-1}$ and thus, using the argument of the previous paragraph we get $@t_n[\Box\,\Diamond\,\neg\varphi] \in \Gamma_n$. But $@t_n[\Box\,\Diamond\,\neg\varphi]$ is inconsistent with $@t_n[\Box\,\varphi]$ so $@t_n[\Box\,\varphi] \notin \Gamma_n$. ■

Constructing a Model from γ

We now want to construct a $\mathcal{M}$ model satisfying γ. This will be done by first building a set of single history models corresponding to branches of the multiple history model and then joining these models together.

For technical reasoning we shall start by considering formula sets that are only maximal with respect to a subvocabulary of the language of $\mathcal{M}$. Specifically, we shall set aside a denumerable set $D = \{\ldots, \delta_i, \ldots\}$ of propositional variables of the atemporal base language $\mathcal{L}_B$ which do not occur in γ. (Later these will be to construct unique indices for the branches of the model). Let $\mathcal{M}^\flat$ be the sub-language of $\mathcal{M}$ consisting of the formulae in $\{\varphi \mid \varphi \in \mathcal{M}$ and no $\delta_i \in D$ occurs in $\varphi\}$. An $\mathcal{M}^\flat$*-maximal consistent* formula set is one which is maximal consistent with respect to the sub-language $\mathcal{M}^\flat$.

We start by constructing Γ_0, a $\mathcal{M}^\flat$-maximal consistent TME-set containing γ (by hypothesis, γ is consistent, so this will always be possible). We then let $\mathfrak{S}$ be the set of all alternative sequences derived from Γ_0. From each $\Sigma_i = \langle\langle\emptyset, \emptyset, \Gamma_0\rangle, \langle t_1, \alpha_1, \Gamma_1\rangle, \ldots, \langle t_n, \alpha_n, \Gamma_n\rangle\rangle$ in $\mathfrak{S}$ we now create a single history model, $\mathfrak{A}^\flat_{\Sigma_i} = \langle T_{\Sigma_i}, \preceq_{\Sigma_i}, B^\flat_{\Sigma_i}, V_{\Sigma_i}, \tau_{\Sigma_i}, \{\mathfrak{h}^\flat_{\Sigma_i}\}\rangle$. This is derived from Γ_n in accordance with the construction given in the $\mathcal{L}$ completeness proof.

Note, in particular that, since $\gamma \in \Gamma_0$, we know that γ is true at some point on one of the generated single history models.

The presence of modal formulae in Γ_n does not affect this construction (although the sets $\Gamma_{n\mathfrak{t}_i}$ associated with each time point $\mathfrak{t}_i$ will now contain modal formulae). Moreover, the same reasoning as used for the $\mathcal{L}$ models shows that each $\Gamma_{n\mathfrak{t}_i}$ will be a maximal consistent subset of $\mathcal{M}^\flat$.

'Tagging' the Single History Models Later in the proof we shall need to ensure that the constructed model does not contain any 'accidentally' confluent histories. We want the only confluent histories to be ones that were forced to be confluent by the method of construction.

Recall that the sets Γ_i do not contain any propositional variables in the set D, so the models $B^\flat_{\Sigma_i}$ will not assign to these variables. We shall now extend each $\mathfrak{A}^\flat_{\Sigma_n}$ model to another single history model $\mathfrak{A}_{\Sigma_n}$, within which each sequence Σ_i, has a unique 'tag' specified by means of an assignment to the variables in D. We note that the set $\mathfrak{S}$ of sequences and $\wp(D)$ (the power set of the set D of 'spare' propositional constants) are both have the cardinality of $\mathbb{R}$. This means that there is a one-to-one mapping $\omega : \mathfrak{S} \rightarrow \wp(D)$, which we can use to associate with each $\Sigma_i \in \mathfrak{S}$ a unique assignment of truth values to the propositions in D. This assignment will be referred to as $\omega(\Sigma_i)$.

From each $\mathfrak{A}^\flat_{\Sigma_i}$ we construct the single history model $\mathfrak{A}_{\Sigma_i} = \langle T_{\Sigma_i}, \preceq_{\Sigma_i}, B_{\Sigma_i}, V_{\Sigma_i}, \tau_{\Sigma_i}, \{\mathfrak{h}_{\Sigma_i}\}\rangle$, which augments $\mathfrak{A}^\flat_{\Sigma_i}$ by adding assignments to the propositional variables in D. In the case of $\Sigma_0 = \langle\langle\emptyset, \emptyset, \Gamma_0\rangle\rangle$, for each $\mathfrak{t} \in T_{\Sigma_0}$ we set $\mathfrak{h}_{\Sigma_0}(\mathfrak{t}) = \mathfrak{h}^\flat_{\Sigma_0}(\mathfrak{t}) \oplus \omega(\Sigma_0)$, where the $\oplus$ operator means that we simply combine the assignment of the base-model $\mathfrak{h}^\flat_{\Sigma_0}(\mathfrak{t})$ with the assignment $\omega(\Sigma_0)$ to the variables of D. We set B_{Σ_i} equal to the range of $\mathfrak{h}_{\Sigma_0}$. When augmenting the other $\mathfrak{A}^\flat_{\Sigma_i}$ models, the construction proceeds in order of increasing length of the generating sequences Σ_i, so that when we come to augment $\mathfrak{A}^\flat_{\Sigma_i}$ we have already augmented those models generated from its sub-sequences.

For each $\Sigma_i = \langle\langle\emptyset, \emptyset, \Gamma_0\rangle, \langle t_1, \alpha_1, \Gamma_1\rangle, \ldots, \langle t_n, \alpha_n, \Gamma_n\rangle\rangle$ the history function $\mathfrak{h}_{\Sigma_i}$ is specified as follows:

- If $\mathfrak{t} \preceq_{\Sigma_i} \tau_{\Sigma_i}(t_n)$ then $\mathfrak{h}_{\Sigma_i}(\mathfrak{t}) = \mathfrak{h}_{\Sigma_j}(\mathfrak{t})$, where Σ_j is the initial sub-sequence of Σ_j omitting just its final element.
- If $\mathfrak{t} \not\preceq_{\Sigma_i} \tau_{\Sigma_i}(t_n)$ then $\mathfrak{h}_{\Sigma_i}(\mathfrak{t}) = \mathfrak{h}^\flat_{\Sigma_0}(\mathfrak{t}) \oplus \omega(\Sigma_i)$.

This specification ensures that $\mathfrak{h}_{\Sigma_i}$ agrees with $\mathfrak{h}_{\Sigma_j}$ up to the point at which $\mathfrak{h}_{\Sigma_i}$ branches off as an alternative history; and for all times after that point it satisfies a Boolean combination (given by $\omega(\Sigma_i)$) of the variables of D that is not satisfied anywhere on any previously generated history.

Finally for each $\mathfrak{A}_{\Sigma_i}$ we again set B_{Σ_i} equal to the range of $\mathfrak{h}_{\Sigma_i}$.

Notation A single history model generated in this way, whose history function is $\mathfrak{h}$, will be referred to as $\mathfrak{A}_\mathfrak{h}$ (the assignments to the D variables guarantee that there is a unique history for each model and *vice versa*), the set of formulae occurring in the final tuple of the sequence from which it was generated will be $\Gamma_\mathfrak{h}$, and the set of formulae associated with time point $\mathfrak{t}$ during the construction of $\mathfrak{A}_\mathfrak{h}$ (according to the specification in section) will be $\Gamma_{\mathfrak{h}\mathfrak{t}}$. Following the notation used for the single history models I let $\Delta_{\mathfrak{h}\mathfrak{t}} = \{\varphi \mid \varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}} \wedge \varphi \in \mathcal{L}_B\}$ and $\mathfrak{B}_{\mathfrak{h}\mathfrak{t}}$ be an $\mathcal{L}_B$ model satisfying $\Delta_{\mathfrak{h}\mathfrak{t}}$.

Joining the Single History Models We now combine all the single history models $\mathfrak{A}_{\Sigma_i}$ to form a multiple history model

$$\mathfrak{M}_\Gamma = \langle T_\Gamma, \preceq_\Gamma, B_\Gamma, V_\Gamma, \tau_\Gamma, H_\Gamma \rangle .$$

From **A21** and **A11** we can prove that $t \leq t' \rightarrow \forall u[@u[\Box(t \leq t')]$; and from **A6** we can derive $t = t' \rightarrow \forall u[@u[\Box(t = t')]]$. So, from the definition of the formula sets Γ_i occurring in the sequences in $\mathfrak{S}$, we see that all single history models Σ_i will agree on T_{Σ_i}, $\preceq_{\Sigma_i}$, V_{Σ_i} and τ_{Σ_i}. Thus we chose any Σ_i and set:

- $T_\Gamma = T_{\Sigma_i}$,
- $\preceq_\Gamma = \preceq_{\Sigma_i}$,
- $V_\Gamma = V_{\Sigma_i}$ and $\tau_\Gamma = \tau_{\Sigma_i}$,
- $B_\Gamma = \bigcup\{B_{\Sigma_i} \mid \Sigma_i \in \mathfrak{S}\}$ (Since the values of B_{Σ_i} will vary between the different Σ_i we take the union of all these values),
- $H_\Gamma = \{\mathfrak{h}_{\Sigma_i} \mid \Sigma_i \in \mathfrak{S}\}$.

γ is true at some point in $\mathfrak{M}_\Gamma$

As was done in the completeness proof for $\mathcal{L}$, I will now show that in the model $\mathfrak{M}_\Gamma$ generated from Γ, every formula $\varphi \in \Gamma$ (including in particular γ) is true at some point in the model (i.e. at some time in one of the single history submodels contained within $\mathfrak{M}_\Gamma$. As with $\mathcal{L}$, this is again done by proving the following more general lemmaa:

Lemma 11 *For all* $\varphi \in \mathcal{M}$, $\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ *iff* $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{T}$.

Proof: The structure of the single history model $\mathfrak{A}_\mathfrak{h}$ is derived entirely from the non-modal formulae of $\Gamma_\mathfrak{h}$. Moreover, it is easy to see that the non-modal formulae of the set $\Gamma_{\mathfrak{h}\mathfrak{t}}$ (associated with time $\mathfrak{t}$ in $\mathfrak{A}_\mathfrak{h}$) is the same set that would be associated with $\mathfrak{t}$ if we built a canonical single history model just from the non-modal formulae of $\Gamma_\mathfrak{h}$. Thus we can appeal to Lemma 5 to prove that for any non-modal formula $\psi \in \mathcal{M}^\flat$, we have $\psi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ if and only if $[\![\psi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}_\Gamma} = \mathbf{T}$.

We must now consider formulae including the $\Box$ operator (we may assume $\Diamond$ has been eliminated by its definition). We precede by induction so that when we consider whether the lemma holds for $\Box\varphi$ we can assume that it has already been demonstrated to hold for φ. We consider an exhaustive set of possible forms of the formula $\Box\varphi$ in terms of the immediate sub-formula φ:

Case of $\Box\varphi$, where φ is Basic: Suppose $\Box\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$. Then by **A17** $\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$. Since φ is basic it is a member of $\Gamma_{\mathfrak{h}\mathfrak{t}}$ just in case $[\![\varphi]\!]_{\mathfrak{B}_{\mathfrak{h}\mathfrak{t}}} = \mathbf{T}$, where $\mathfrak{B}_{\mathfrak{h}\mathfrak{t}} = \mathfrak{h}(\mathfrak{t})$. But for any history $\mathfrak{h}'$ such that $\mathfrak{h}' \stackrel{\mathfrak{t}}{\approx} \mathfrak{h}$ we know that $\mathfrak{h}'(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t})$; so $[\![\varphi]\!]_{\mathfrak{B}_{\mathfrak{h}\mathfrak{t}}} = [\![\varphi]\!]_{\mathfrak{B}_{\mathfrak{h}'\mathfrak{t}}}$. By the induction hypothesis $\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ iff $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{T}$; so φ must be true at $\mathfrak{t}$ for all histories which are alternatives to $\mathfrak{h}$ at $\mathfrak{t}$. Hence, by **S9**, we must have $[\![\Box\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{T}$. On the other hand, if $\Box\varphi \notin \Gamma_{\mathfrak{h}\mathfrak{t}}$ then, because of maximal consistency, $\neg\Box\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ and then from **A19** we also have $\neg\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$, so $\varphi \notin \Gamma_{\mathfrak{h}\mathfrak{t}}$. Thus by the induction hypothesis $[\![\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{F}$ and so (since $\mathfrak{h} \stackrel{\mathfrak{t}}{\approx} \mathfrak{h}$) from **S9** we get $[\![\Box\varphi]\!]^{\mathfrak{h}\mathfrak{t}}_{\mathfrak{M}} = \mathbf{F}$.

Case of $\Box(t \leq t')$: To demonstrate this I show that if $t \leq t' \in \Gamma_{\mathfrak{h},\mathfrak{t}}$ then $t \leq t'$ is a member of every formula set $\Gamma_{\mathfrak{h}_i\mathfrak{t}_i}$. From **A11** $t \leq t' \leftrightarrow @t''[(t \leq t')]$, for any t''. Then using **A7** and **A21** we see that $t \leq t' \rightarrow @t''[\Box(t \leq t')]$. Using this together with Lemma 10 we see that $t \leq t'$ must be a member in every formula set $\Gamma_{\mathfrak{h}_i}$ and thus so must $@t''[\Box(t \leq t')]$ for any t''. By the way the single history sets are constructed from each $\Gamma_{\mathfrak{h}_i}$ we then see that $t \leq t'$ is indeed a member of every $\Gamma_{\mathfrak{h}_i\mathfrak{t}_i}$. Given the universality of such formulae it is easy to establish this case of the induction step.

Case of $\Box@(t)$: Using **A20**, **A14** and **A13** it can be shown that if $@(t)$ holds at any time point in any history it must hold at that point in any history. Thus, clearly, if $\Box@(t)$ holds at some time point in any history, $@(t)$ must hold at that time in all confluent histories.

Case of $\Box\Box\varphi$: From the $S5$ axioms $\Box\Box\varphi \leftrightarrow \Box\varphi$, so $\Box\Box\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ just in case $\Box\varphi \in \Gamma_{\mathfrak{h}\mathfrak{t}}$. So we can reduce this case to one of the others.

Case of $\Box\forall t[\varphi]$: By **TBF** $\Box\forall t[\varphi]$ is equivalent to $\forall t[\Box\varphi]$. By the induction hypothesis, the lemma must hold for $\Box\varphi$. To get the induction for $\forall t[\Box\varphi]$ we can then use the same reasoning used for time quantification in the non-modal logic $\mathcal{L}$.

Case of $\Box@t[\varphi]$: This is the key case. We need to show that if $\Box@t[\varphi] \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ then $@t[\varphi] \in \Gamma_{\mathfrak{h}'\mathfrak{t}}$ for all histories $\mathfrak{h}'$ that are confluent with $\mathfrak{h}$ up to $\mathfrak{t}$. We shall show this by performing an induction over all histories and show that each one satisfies the condition: either $\Box@t[\varphi] \in \Gamma_{\mathfrak{h}'\mathfrak{t}}$ or $\mathfrak{h}'$ is not confluent with $\mathfrak{h}$ up to $\mathfrak{t}$. The induction will we be specified relative to the set of sequences Σ_i from which the histories are constructed. Starting with any sequence $\Sigma_\mathfrak{h} = \langle \ldots, \langle t', \alpha, \Gamma_\mathfrak{h}\rangle\rangle$ we can get to any other sequence by a series of truncations and or extensions by alternatives.

Clearly the starting point satisfies the conditions since we know that $\Box@t[\varphi] \in \Gamma_{\mathfrak{h}\mathfrak{t}}$ and hence by **A17** $@t[\varphi] \in \Gamma_{\mathfrak{h}\mathfrak{t}}$. We just need to show that truncations and legitimate extensions preserve the desired condition.

The coding with Boolean combinations of the formulae in D means that from the point $\tau_\Gamma(t)$ onwards the history $\mathfrak{h}$ generated from $\Sigma_\mathfrak{h}$ must be distinct from any history $\mathfrak{h}'$ generated by any extension or truncation of Σ_i. Since all formulae β of the atemporal sub-language $\mathcal{L}_B$ satisfy **A19**, we can now use **A22** to show that for any time prior to $\tau_\Gamma(t)$ if any such β is true on $\mathfrak{h}$ it is also true on $\mathfrak{h}'$. Consequently $\mathfrak{h}$ and $\mathfrak{h}'$ must be confluent up to the time $\tau_\Gamma(t)$. This means that all histories confluent with $\mathfrak{h}$ up to $\tau_\Gamma(t)$ can be reached by a series of truncations and extensions of Σ_i such that the time variable t_i in the last tuple of each sequence in the series always satisfies $\tau_\Gamma(t) \preceq_\Gamma \tau_\Gamma(t_i)$. To complete the proof we just need to show that for any such history $\mathfrak{h}'$ we have $@t[\varphi] \in \Gamma_{\mathfrak{h}'\mathfrak{t}}$.

Suppose $\Box @t[\varphi]$ is in some arbitrary $\Gamma_{\mathfrak{g}\mathfrak{t}_i}$ on the structure. Then, $\Box\Box @t[\varphi] \in \Gamma_{\mathfrak{g}\mathfrak{t}_i}$ and also $@t'[\Box\Box @t[\varphi]]$ for some variable t'. This means that any $\Gamma_{\mathfrak{g}'}$ that is a $t\alpha$-alternative to $\Gamma_{\mathfrak{g}}$ must contain the formula $\Box @t[\varphi]$. Suppose that $t \leq t'$ (as we have seen, if this is in any history it is true for all), then by **A22** we can derive that $@t'[\Box @t[\varphi]] \in \Gamma_{\mathfrak{g}'}$. Then using **A17**, **A7** and **A10** we get $@t[\varphi] \in \Gamma_{\mathfrak{g}'}$ as required.

I have demonstrated the induction step for all formulae $\Box\varphi$ so the lemma is proved. ■

Theorem 2 *The axiom system for $\mathcal{M}$ is complete with respect to the given semantics.*

Proof: We need to show that any consistent formula has a model. I have shown that any consistent formula γ is a member of a maximal consistent TME-set Γ and from this we can build a canonical model $\mathfrak{M}_\Gamma$ such that γ must be true at some time on some history within this model. ■

Conclusion

I have given an axiomatisation of a very expressive logic of branching histories, which I believe to be useful in formalising commonsense concepts which relate to time and possibility. This forms a significant fragment of the logic presented in (Bennett & Galton to appear). I have found that the system is capable of describing many aspects of physical processes and one can define within it various causal relationships. Probably the logic is to complex to serve as a vehicle for effective reasoning. Rather it is intended to provide a general framework within which more limited, application-oriented,formalisms could be embedded.

More work remains to be done tightening up some of the details of the proof.

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