Hybrid Partition Machines with Disturbances

Ekaterina S. Lemch and **Peter E. Caines**
Department of Electrical and Computer Engineering,
McGill University, 3480 University Street,
Montreal, Quebec, Canada, H3A 2A7,
⟨lemch,peterc⟩@cim.mcgill.edu

Abstract

The notions of finite analytic partition, dynamical consistency, and partition machine were originally developed in (Wei 1995; Caines & Wei 1998) for dynamical systems on differentiable manifolds and in (Caines & Lemch 1999) for hybrid systems; in this paper they are generalized to hybrid systems with disturbances.

1. Introduction

Many complex control systems in engineering, such as air traffic management systems (Tomlin, Pappas, & Sastry 1998), motor drives (Balluchi *et al.* 1998) and intelligent highway systems (Lygeros, Godbole, & Sastry 1998), have a hybrid nature, in the sense that (i) at the lowest level they can be characterized by continuous differential equations, (ii) at the highest level by a discrete mechanism, and (iii) the evolution of the overall system is described by the interaction of all system levels.

In this paper a hybrid (base) system is modelled as a quintuple consisting of a state space (which is the direct product of a set of discrete states and an n-dimensional manifold), sets of admissible continuous and discrete controls, a family of controlled autonomous vector fields assigned to each discrete state, and a (partially defined) map of discrete transitions. Next, generalizing the theory presented in (Caines & Wei 1998), the notion of a finite analytic partition Π of a state space of a hybrid system is defined. Then the notion of dynamical consistency is generalized to that of *hybrid dynamical consistency*. Based on these notions, the *partition machine* H^{Π} of a hybrid system H is defined in such a way that, in the class of *in-block controllable partitions*, the controllability of the high level system (described by the partition machine, which is a discrete finite state machine) is equivalent (under some technical conditions) to the controllability of the low level system (described by differential equations). Within the *hybrid partition machine* framework, a discrete controller supervises its continuous subsystems via hierarchical feedback relations; furthermore, each continuous subsystem is itself (internally) subject to feedback control. The resulting hierarchical control structure is such that the high level controller makes supervisory decisions, while the low level controllers will actually generate the required state trajectories.

An important problem within this framework is that of finding conditions for a partition to satisfy the in-block controllability hypothesis. In Section 4, a form of local accessibility for nonlinear control systems is introduced called the *continuous fountain condition*. To verify the in-block controllability hypothesis using the theory of this paper it is sufficient to establish that (i) the continuous fountain condition and (ii) a recurrence condition hold. Furthermore, for the so-called energy slice partitions of Hamiltonian systems (and, in fact, partitions of more general systems), the dense recurrence condition under a distinguished control is an inherent property which does not need explicit verification (whenever each slice is precompact).

In Section 5, an application of the theory to a highly simplified air traffic management system is presented.

2. Hybrid Partition Machines for Hybrid Systems

Consider a *hybrid system* H which, in this paper, is taken to be the quintuple

$$H = \{L \triangleq Q \times M, U, \Sigma, f, \Gamma\}, \tag{1}$$

where
$Q = \{q_1, \cdots, q_m\}$ is a set of discrete states (which are called *control locations*);
M is an open connected subset of $\Re^n$;
L is called the *state space of* H;
$U \subset \Re^u$ is a set of admissible controls;
Σ is a finite set of transition labels;
$f : Q \times M \times U \to TM$ is an autonomous vector field assigned to each control location;
$\Gamma : L \times \Sigma \to L$ is an untimed (partially defined) map of discrete transitions.

Similar models of hybrid systems are employed by (Branicky, Borkar, & Mitter 1994; Broucke 1998; Lygeros, Godbole, & Sastry 1998; Tomlin, Pappas, & Sastry 1998) and others.

Henceforth in this paper the notation

$$\Gamma((q_1, y_1); \sigma(t^*)) \equiv \Gamma((q_1, y_1); \sigma) = (q_2, y_2)$$

(where $q_1, q_2 \in Q, y_1, y_2 \in M, \sigma \in \Sigma$) means that for any specified instant $t^* \in \Re$ and control σ,

(i) $\lim_{t \to t^{*-}} q_t = q_1, \lim_{t \to t^{*+}} q_t = q_{t^*} = q_2$;

(ii) $\lim_{t \to t^{*-}} y(t) = y_1, \lim_{t \to t^{*+}} y(t) = y(t^*) = y_2$.

According to the description above, the continuous and discrete state trajectories are continuous from the right. In fact the discrete state is piece-wise constant from the right.

The symbol Σ shall denote the original set Σ extended with the identity element $\{id\}$.

It shall be assumed that the system H is *non-zeno* in the sense that the class of inputs $\{\sigma\}$ is restricted so that in finite time only a finite number of discrete transitions may be generated.

The continuous dynamics of the hybrid system at each control location $q \in Q$ are described by a differential equation of the form

$$S^q: \quad \dot{x} = f(q, x, u), \qquad x \in M, u \in U, \qquad (2)$$

for which the assumptions are:

(i) U is the sets of all bounded piece-wise $C^k(\Re^1)$ ($k \geq 1$) functions of time continuous from the right (i.e. functions which are k times continuously differentiable, except possibly at a finite number of points in any compact interval, and which are bounded on any bounded subset and everywhere continuous from the right);

(ii) for each $q \in Q$, $f(q, \cdot, \cdot)$ is continuously differentiable with respect to its arguments $x \in M$ and $u \in U$;

(iii) M is a non-empty, open, connected subset of $\Re^n$.

The symbol $\phi_q(t, x, u)$ shall denote the transition function of the vector field $f(q, x, u)$ with respect to the initial condition $x \in M$ and control $u \in U$.

Generalizing (Wei 1995; Caines & Wei 1998) to the case $card(Q) > 1$, we formulate the notion of a cellular decomposition of $Q \times M$ given below. We note that the definition implies the set M has certain additional properties and *hencerforth in this paper we assume this to be the case.*

Definition 1 A *finite analytic partition of the state space* $L = Q \times M$ of a hybrid system H is a collection $\Pi = \{\pi_q; q \in Q\}$ where each π_q, $q \in Q$, is a finite analytic partition of M. By definition a finite analytic partition π_q is a pairwise disjoint collection of subsets $\pi_q = \{X_1^q, X_2^q, \cdots, X_{k_q}^q\}$ such that every X_i^q is non-empty, open, path-connected, and is such that

$$\bar{M} = \bigcup_{i=1}^{k_q} (X_i^q \cup \partial X_i^q),$$

where, further, the boundary ∂X_i^q of every block X_i^q is a locally finite union of connected components of $n - p$ dimensional, $p \geq 1$, analytic manifolds (possibly with boundary). □

Definition 2 Let $p = (q, x) \in L$, $p' = (q', x') \in L$ and let $R \subset L$. We shall say that p' *is accessible from* p *with respect to* R if there exists a finite sequence of admissible control pairs

$$\{u(t), t \in [T_0, T_1); \sigma(T_1)\},$$
$$\cdots$$
$$\{u(t), t \in [T_{k-1}, T_k); \sigma(T_k)\},$$

such that the state trajectory $p(\cdot) = (q(\cdot), x(\cdot))$ defined as follows

$$\begin{aligned} p(T_0) &= p; \\ p(t) &= \ (q(T_i), \phi_{q(T_i)}(t, x(T_i), u)), \\ &\qquad t \in (T_i, T_{i+1}), 0 \leq i \leq k-1; \\ p(T_{i+1}) &= \Gamma(\lim_{t \to T_{i+1}^-} p(t); \sigma(T_{i+1})) \end{aligned}$$

satisfies (i) $p(t) \in R$, for all $t \in [T_0, T_k]$; and (ii) $p(T_k) = p'$.

The set of all states accessible from p (with respect to $R \subset L$) shall be denoted by $A^R(p)$.

The state space L of H is said to be *controllable* if $A^L(p) = L$, for all $p \in L$. □

Definition 3 Let $q_1, q_2 \in Q$ and let $X^{q_1} \in \pi_{q_1} \in \Pi$, $Y^{q_2} \in \pi_{q_2} \in \Pi$, where Π is a finite analytic partition of the state space L of H. $\langle X^{q_1}, Y^{q_2} \rangle$ is said to be a *hybrid dynamical consistent pair* (HDC) if and only if for any $x \in X^{q_1}$ there exist a state $x' \in X^{q_1}$ such that (q_1, x') is accessible from (q_1, x) with respect to $q_1 \times X^{q_1} \subset L$ and at least one of the following conditions holds:

(a) there exists a discrete control $\sigma \in \Sigma$ such that $\Gamma((q_1, x'); \sigma) = (q_2, y)$, for some $y \in Y^{q_2}$

OR

(b) $q_2 = q_1$, i.e. $X^{q_1}, Y^{q_2} \in \pi_{q_1}$, and there exists a control $u \in U$ defined on a finite time interval $[0, T]$ such that

(i) $\phi_{q_1}(t, x', u) \in X^{q_1}$, for all $t \in [0, t^*)$, $0 < t^* < T$;

(ii) $\phi_{q_1}(t^*, x', u)$ is a facial boundary state of $\partial X^{q_1} \cap \partial Y^{q_2}$, i.e. it lies in the relative interior of an $(n-1)$ dimensional connected component of the boundaries ∂X^{q_1} and ∂Y^{q_2}; and

(iii) $\phi_{q_1}(t, x', u) \in Y^{q_2}$, for all $t \in (t^*, T]$. □

The notion of hybrid dynamical consistency for hybrid systems is evidently a generalization of the notion of *dynamical consistency (DC)* introduced in (Caines & Wei 1998) for control systems on differentiable manifolds.

Definition 4 Let H be a hybrid system

$$H = \{L \triangleq Q \times M, U, \Sigma, f, \Gamma\}$$

and $\Pi = \{\pi_q; q \in Q\} = \{\{X_1^q, \cdots, X_{k_q}^q\}; q \in Q\}$ be a finite analytic partition of L. Consider the system $H^\Pi = \{L^\Pi, \bar{E}, \Phi^\Pi\}$, where
L^Π, which is called the *state space of* H^Π, is defined as

$$L^\Pi \triangleq \{(q, i); q \in Q, i \in \{1, \cdots, k_q\}\};$$

$\bar{E} = \{E^{q_2,j}_{q_1,i}; q_1, q_2 \in Q, 1 \leq i \leq k_{q_1}, 1 \leq j \leq k_{q_2}\}$ is a finite set of transition labels;

$\Phi^{\Pi} : L^{\Pi} \times \bar{E} \to L^{\Pi}$ is a (partially defined) map such that $\Phi^{\Pi}((q_1,i); E^{q_2,j}_{q_1,i}) = (q_2,j)$ if and only if $\langle X^{q_1}_i, X^{q_2}_j \rangle$ is a hybrid dynamical consistent pair. Otherwise $\Phi^{\Pi}((q_1,i); E^{q_2,j}_{q_1,i})$ is not defined.

The finite state machine H^{Π} is called the *hybrid partition machine* of H. □

Definition 5 Let Π be a finite analytic partition of the state space L of H. Π is called *hybrid in-block controllable* if for any $X^q \in \pi_q \in \Pi$, $q \in Q$, and any $x \in X^q$, the set of states accessible from (q,x) with respect to $q \times X^q \subset L$ is equal to $q \times X^q$. □

Theorem 1 Let Π be a hybrid in-block controllable partition of L. Then

H^{Π} is controllable (as a finite state machine)

$\Longleftrightarrow$

$L^* \triangleq \bigcup_{q \in Q} q \times (M - \partial\pi_q)$ is controllable with respect to $\tilde{L} \triangleq \bigcup_{q \in Q} q \times (M - NF(\pi_q))$, where $NF(\pi_q)$, $q \in Q$, is the set of all non-facial states of the partition π_q. □

3. Hybrid Systems with Disturbances

A *hybrid system with disturbances*, denoted $H + D$, is the quintuple

$$H = \{L \triangleq Q \times M, \mathcal{U} \triangleq \mathcal{U}_c \times \mathcal{U}_d, \Sigma \triangleq \Sigma_c \cup \Sigma_d, f, \Gamma\},$$

where

(i) for each $q \in Q$, $f_q : M \times \mathcal{U}_c \times \mathcal{U}_d \to TM$ is continuously differentiable with respect to its arguments;

(ii) Q and M are as defined in the previous section;

(iii) $\mathcal{U}_d$ is a class of disturbances which in this paper is taken to be the set of all functions $v : \Re \to \Re^{m_v}$ which are

  (a) $C^1(\Re^1)$ functions of time except (possibly) at each element of a sequence $\{t_i\}$ which does not have an accumulation point,

  (b) continuous from the right at any point of discontinuity t_i,

  (c) bounded on any bounded subset of $\Re$;

(iv) $\mathcal{U}_c$ is a class of admissible disturbance and state feedback control functions $u : \Re \times \Re^{m_v} \times L \to \Re^{m_u}$ such that

  (a) u is bounded on any bounded subset of $\Re \times \Re^{m_v} \times q \times M$, for each $q \in Q$,

  (b) $u(t,v,p)$ is continuously differentiable in (t,v,p) with a bounded differential on any bounded subset of $\Re \times \Re^{m_v} \times q \times M$, for each $q \in Q$, except (possibly) at each element of a sequence $\{t_i\}$ which does not have an accumulation point,

  (c) at any such t_i, $u(t,v,p)$ is jointly continuously differentiable with respect to t from the right;

(v) Σ_c and Σ_d denote the sets of discrete (state and disturbance) feedback controlled transitions and discrete disturbance transitions, respectively. At any instant t^* of application of a discrete feedback controlled transition σ_{t^*} it is the case that $\sigma_{t^*} = \sigma_{t^*}(p_{t^*_-}, v_{t^*_-})$.

We define the *disturbance co-accessible set* $CA^{dis}(K;R)$ to $K \subset L$ with respect to $R \subset L$ to be

$$CA^{dis}(K;R) = \bigcup_{n=1}^{\infty} CA^{dis}_n(K;R),$$

where $CA^{dis}_n(K;R) = CA^{dis}_1(CA^{dis}_{n-1}(K;R);R)$, for $n > 1$, and, finally,

$$\begin{aligned} CA^{dis}_1(K;R) = \{(q,x');\ & \exists v \in \mathcal{U}_d\ \exists T > 0\ \forall u \in \mathcal{U} \\ & \phi_q(T,x',u,v) \in K, \text{ and} \\ & \forall t \in [0,T]\ \phi_q(t,x',u,v) \in R\} \end{aligned}$$

$$\bigcup \{(q',x') \in R;\ \exists \mu \in \Sigma_d\ \Gamma((q',x');\mu) \in K\}.$$

Let $X, Y \in \Pi$, where Π is a finite analytic partition of the state space L of H. Define

(i) $R = X \cup Y \cup Facial(\partial X \cap \partial Y)$, if $X, Y \in \pi_q \in \Pi$, for some $q \in Q$; and

(ii) $R = X \cup Y$, otherwise (i.e. if $X \in \pi_{q_1} \in \Pi$, $Y \in \pi_{q_2} \in \Pi$, $q_1, q_2 \in Q$ and $q_1 \neq q_2$).

Definition 6 We shall say that the *(undisablable) disturbance event* D^Y_X is defined if there exists $x \in X$ such that $x \in CA^{dis}(Y;R)$ □

We define the *disturbance rejecting co-accessible set* $CA^{dr}(K;R)$ to $K \subset R$ with respect to $R \subset L$ to be

$$\bigcup_{n=1}^{\infty} CA^{dr}_n(K;R),$$

where $CA^{dr}_n(K;R) = CA^{dr}_1(CA^{dr}_{n-1}(K;R);R)$, for $n > 1$, and, finally,

$$\begin{aligned} CA^{dr}_1(K;R) = \{(q,x');\ & \forall v \in \mathcal{U}_d\ \forall T > 0 \\ & \exists u \in \mathcal{U} \text{ defined on } [0,T] \\ & \phi_q(T,x',u,v) \in K, \text{ and} \\ & \forall t \in [0,T]\ \phi_q(t,x',u,v) \in R\} \end{aligned}$$

$$\bigcup \{(q',x') \in R;\ \exists \sigma \in \Sigma_c\ \Gamma((q',x');\mu) \in K\}.$$

Definition 7 Let $X, Y \in \Pi$, where Π is a finite analytic partition of the state space L of H. We shall say that the *(disturbance rejecting) control event* U^Y_X is defined if $x \in CA^{dr}(Y;R)$, for all $x \in X$. □

Remark: Controlled and disturbance events are mutually exclusive in the sense that if D_X^Y for some partition blocks $X, Y \epsilon \Pi$, then $\neg U_X^Z$, for any $Z \epsilon \Pi$, $Z \neq Y$.

Definition 8 A *hybrid partition machine with disturbances*, denoted HPM+D, is a finite state machine $H^\Pi = \{L^\Pi, \{U_{q_1,i}^{q_2,j}\}, \{D_{q_1,i}^{q_2,j}\}\}$ where the state space $L^\Pi = \{(q,i); q \epsilon Q, i \epsilon \{1, \cdots, k_q\}\}$, and $D_{q_1,i}^{q_2,j}$ and $U_{q_1,i}^{q_2,j}$ are the disturbance and control events, respectively. □

Definition 9 A block $X \epsilon \Pi$ is called *(disturbance rejecting) in-block controllable*, denoted DR-IBC, if for all $p, p' \epsilon X$, $p' \epsilon CA^{dr}(p; X)$. □

Definition 10 A block $X \epsilon \Pi$ is called *ε-DR-IBC* if
(i) $E_\varepsilon \triangleq \{p \epsilon X; N_\varepsilon(p) \subset X\} \neq \emptyset$, and
(ii) for all $p \epsilon X$ and any $p' \epsilon E_\varepsilon$, $p \epsilon CA^{dr}(N_\varepsilon(p'); X)$.

□

Let $F \subset L$ denote the set of *forbidden states.* We shall say that there exists a *safe low level feedback control law* for $p, p' \epsilon L$, if $p \epsilon CA^{dr}(p'; L - F)$. Furthermore, we shall say that there exists an *ε-safe low level feedback control law* for $p, p' \epsilon L$, if $p \epsilon CA^{dr}(N_\varepsilon(p'); L - F)$. It may be shown that

Theorem 2 (i) Let $X, Y \epsilon \Pi$ and let Y be DR-IBC. If there exists a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, such that U_i^{i+1} is defined for any two consecutive blocks in this sequence, then, for any $x \epsilon X$, $y \epsilon Y$, there exists a safe low level feedback control.
(ii) Assume that there exists a safe low level feedback control law for some $p, p' \epsilon L$, and assume further that a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, $p \epsilon X$, $p' \epsilon Y$ contains the resulting trajectories for all $v \epsilon \mathcal{U}_d$ and is such that each of Z_i $(0 \leq i \leq r)$ is DR-IBC. Then the sequence $\{U_i^{i+1};\ 0 \leq i \leq r-1\}$ of disturbance rejecting control events is defined.

□

Theorem 3 (i) Let $X, Y \epsilon \Pi$ and let Y be ε-DR-IBC. If there exists a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, such that U_i^{i+1} is defined for any two consecutive blocks in this sequence, then, for all $x \epsilon X$ and any $y \epsilon Y$ such that $N_\varepsilon(y) \subset Y$, there exists an ε-safe low level feedback control.
(ii) Assume that for some $\varepsilon^* > 0$ there exists an ε^*-safe low level feedback control law for some $p, p' \epsilon L$, and assume further that a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, $p \epsilon X$, $p' \epsilon Y$ contains the resulting trajectories for all $v \epsilon \mathcal{U}_d$ and is such that each of Z_i $(0 \leq i \leq r)$ is ε-DR-IBC, for all $\varepsilon > 0$. Then the sequence $\{U_i^{i+1};\ 0 \leq i \leq r-1\}$ of disturbance rejecting control events is defined.

□

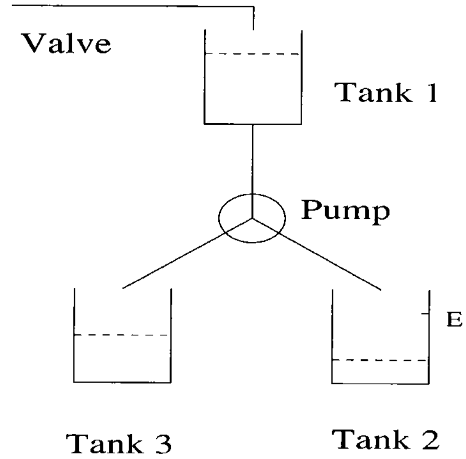

Figure 1: Three tank system.

Example 1 To illustrate the introduced in this paper theory of hybrid systems with disturbances we present the following example.

Consider the system of three tanks shown on Figure 1 (Tittus & Egardt 1998). The system has five different control modes:
q_1: the fluid is pumped from tank 1 to tank 3;
q_2: the fluid is pumped from tank 3 to tank 2;
q_3: the fluid is pumped from tank 2 to tank 1;
q_4: the fluid is added to the system via Valve;
q_5: the pump and the valve are shut off.

The three tanks system can be modelled as a hybrid system in the following way. We shall distinguish five discrete locations - each corresponds to one of the five different control modes, i.e. the set of discrete states is $Q = \{q_1, q_2, q_3, q_4, q_5\}$.

The continuous dynamics at the locations are as follows:

$$\begin{aligned} q_1 &: \dot{x} = [\ -1 \quad 0 \quad 1]^T \\ q_2 &: \dot{x} = [\ 0 \quad 1 \quad -1]^T \\ q_3 &: \dot{x} = [\ w \quad -w \quad 0]^T \\ q_4 &: \dot{x} = [\ v \quad 0 \quad 0]^T \\ q_5 &: \dot{x} = [\ 0 \quad 0 \quad 0]^T \end{aligned},$$

where $v, w \epsilon (0, 1)$. The state space for the system is taken to be

$$D = \{(x_1, x_2, x_3) \epsilon \Re^3;\ x_i > 0, i = 1, 2, 3\}.$$

The forbidden states are

$$F = \{(x_1, x_2, x_3); x_i \not\epsilon [A_i, B_i]\},$$

for some given $0 < A_i < B_i$ $(i = 1, 2, 3)$.

For each $q \epsilon Q$, we construct the partition π_q, $\|\pi_q\| = k_q > 0$, obtained by dividing D by the two-dimensional planes $x_i = -x_j + C_{ij}^\ell$ $(i \neq j)$ and $x_i = C_i^\ell$, where $1 \leq i, j \leq 3$ and C_{ij}^ℓ, C_i^ℓ takes values in $\{0, 1, \cdots, n\}$ Hence each cube $\{(x_1, x_2, x_3) \epsilon \Re^3;\ C_i^\ell \leq x_i \leq C_i^\ell + 1\}$ is divided into eight parts.

The forbidden blocks are any $X \epsilon \Pi$ such that $X \cap F \neq \emptyset$.

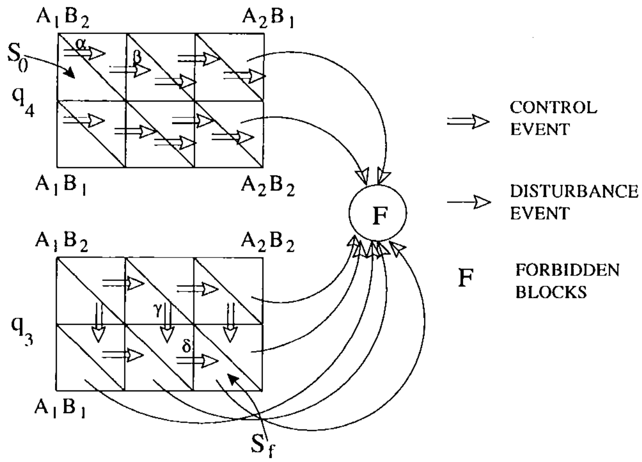

Figure 2: A two-dimensional projection on (x_1, x_2) of the f_{q_3} and f_{q_4} dynamics.

(Note that π is not in the class of hybrid in-block controllable partitions).

In the absence of disturbances (i.e. in the case when $v = v^0 \in (0,1)$, $w = w^0 \in (0,1)$ are known parameters), the existence of the hybrid dynamical consistency relation for a pair of blocks $X, Y \in \pi_q$, such that $Facial(\partial X \cap \partial Y) \neq \emptyset$, can be tested as follows. Let $\bar{n}$ be the normal vector to the boundary $\partial X \cap \partial Y$ oriented in such a way that it points to Y. If the scalar product $\langle \bar{n}, f_q \rangle$ (where f_q is the vector field at the location q) is positive, then the ordered pair $\langle X, Y \rangle$ is hybrid dynamically consistent. Moreover, in the case when v, w are viewed as unknown disturbances, the disturbance rejecting control event U_X^Y is defined.

Further, for each subset $X \in D$ such that $X_i \triangleq q_i \times X \in \pi_{q_i}$, $X_j \triangleq q_j \times X \in \pi_{q_j}$, the control event $U_{X_i}^{X_j}$ is defined (this is because switching between control modes q_i and q_j is allowed at any state).

Assume, in addition, that there is a leak at the second tank at the level $x_2 = E$ $(E < A_2)$. Then, for each block $X \in \pi_{q_i}$ $(1 \leq q_i \leq 5)$ such that $X \cap \{(x_1, x_2, x_3); x_2 > E\} \neq \emptyset$, an undisablable transition is defined to another discrete location $\bar{q}_i$ at which the dynamics of the fluid are described by the differential equation

$$\dot{x} = [a_i \ \ (b_i - 1) \ \ c_i]^T,$$

where $[a_i \ \ b_i \ \ c_i]^T$ is the vector field at the location q_i.

The objective for the three tank system could be formulated in terms of reachability of a finite state or region from an initial state or region while avoiding, if possible, the set of forbidden states and, at the high level, the set of blocks with undisablable transitions leading to forbidden blocks. Using the *partition machines with disturbances* framework, this problem has been reduced to the question of reachability for a finite state machine. As can be seen from Figure 2, the sequence of control modes (together with the interleaved high level control events U_α, U_β, U_γ, U_δ) which drives the three tanks system from the initial state S_0 to the final region S_f can be chosen to be, for instance, q_4, q_4, q_3, q_3, q_0.

□

4. On Construction of IBC Partitions for Systems with First Integrals

This section of the paper concerns the global controllability of nonlinear systems of the form (2). There is an extensive literature on various forms of the accessibility and controllability problems for nonlinear systems. In particular, in (Jurdjevic 1997; Kunita 1979; Lobry 1974), it has been shown that control affine systems satisfying a dense recurrence condition and a full rank Lie algebra condition are (globally) controllable. In (Manikonda & Krishnaprasad 1997), sufficient conditions for controllability of affine nonlinear control systems (where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field) are presented. In (Hammer 1998), an open function condition is employed to prove global reachability.

A state $x \in M$ is said to be a *continuous fountain* if (i) there exists an open ball neighbourhood $B_\rho(x) \subset M$ such that for all open ball neighbourhoods $B_\delta(x)$, where $\delta < \rho$, the accessibility and co-accessibility sets from x relative to $B_\delta(x)$ are open when the state x is deleted; and (ii) whenever (i) holds throughout an open neighbourhood $N(x)$ of x, the limit supremum of the radii ρ of the open ball neighbourhoods of $y \in N(x)$ for which (i) holds is continuous at x.

A state $x \in M$ is called *control recurrent* if it lies in a non-trivial positive limit set under some control $u \in \mathcal{U}$, i.e. x is not an equilibrium point under u and $x = \lim_{n\to\infty} \phi(t_n, x, u)$, for some sequence $\{t_n; n = 1, 2, \cdots\}$ such that $\lim_{n\to\infty} t_n = \infty$.

Theorem 4 *(Caines & Lemch 1998)* Assume that the system S on the open connected state space M is such that every state $x \in M$ is a continuous fountain and through each x in M there exists a non-trivial orbit. Then M is controllable.

□

Theorem 5 *(Caines & Lemch 1998)* Assume that the system S on the open connected state space M is such that every state $x \in M$ is a continuous fountain and for every x in M there exists a state dependent control $u_x(\cdot) \in \mathcal{U}(\Re^n; \Re^m)$ such that x is control recurrent under u_x. Then M is controllable.

□

4.1 The Fountain Condition for Control Affine Systems

Consider a system S of the *input-linear* or *control affine* class of nonlinear time-invariant control systems

$$S: \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \qquad (3)$$

where $f, g_1, g_2, \cdots, g_m$ are smooth mappings from $\Re^n$ into $\Re^n$.

It can be shown that the fountain condition for control affine systems does not imply the Lie Algebra Rank Condition.

Conversely, the Lie Algebra Rank Condition does not imply the fountain condition (Caines & Lemch 1998; 1999).

4.2 Hamiltonian Control Systems

In this section an application of the results of Section 4 to *affine Hamiltonian control systems* is considered. Such systems have a smooth Hamiltonian of the form

$$H(q,p,u) = H_0(q,p) - \sum_{j=1}^{m} H_j(q,p)u_j,$$

where $H_0(q,p)$ is the *internal Hamiltonian (energy)* and H_j, $j = 1, 2, \cdots, m$, are the *interaction* or *coupling* Hamiltonians.

Definition 11 A u_0 *energy slice* $ES(H^-, H^+)$ *of a Hamiltonian control system* is the set of states for which the value of H, under the state dependent control $u_0 \in \mathcal{U}(\Re^n; \Re^m)$, lies between some fixed values H^- and H^+

$$ES(H^-, H^+) \triangleq \{(p,q);\ (p,q) \in E \text{ and } H^- < H(p,q,u)_{|u=u_0} < H^+\}.$$

□

Theorem 6 *(Caines & Lemch 1998)* A Hamiltonian control system for which all states are continuous fountains and all equilibrium points under some constant control $u_0 \in \mathcal{U}$ are isolated is such that any precompact connected component of a u_0 energy slice is controllable.

□

Note that arguments analogous to those used in the proof of Theorem 6 can be applied to any dynamical system for which there is a measure preserving flow and precompact slices based on any (not necessarily energy) first integral of the system motion. The theorem is stated for affine Hamiltonian systems and energy based slices because of their great importance.

5. Application to a Planar Aircraft System

Current air traffic control management is highly centralized; it provides quite detailed instructions to all aircraft regardless of the short term objectives of an individual aircraft (Perry 1997). Air traffic control is an example of a hybrid control system which consists of differential equations coupled with a discrete event system. There is an extensive literature on this subject, see, for example, (Tomlin, Pappas, & Sastry 1998), and the references therein.

Using the approach developed in earlier this paper, we propose a decomposition of the state space of the system, which models the relative motion of two planar aircraft, A_1 and A_2. The motion of an aircraft is described by the left invariant vector field (Tomlin, Pappas, & Sastry 1998)

$$\dot{g} = gX, \tag{4}$$

where

$$g = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix},$$

$$X = \begin{bmatrix} 0 & -\omega & v \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

x, y, ϕ represent the planar position and orientation of the aircraft; v, ω represent the translational and rotational velocities.

Let $g_1(x_1, y_1, \phi_1)$, $g_2(x_2, y_2, \phi_2)$, and $g(x, y, \phi)$ denote the configuration of aircraft A_1, the configuration of aircraft A_2, and the relative configuration of A_2 with respect to A_1, respectively. Then

$$\dot{g}_1 = g_1 X_1, \quad \dot{g}_2 = g_2 X_2, \quad \text{and} \quad g_2 = g_1 g,$$

where X_1 and X_2 are given by the velocity (v_1, ω_1) of A_1 and the velocity (v_2, ω_2) of A_2. The relative motion of two aircraft is described by

$$S_r \quad \begin{cases} \dot{x} = v_2\cos\phi + \omega_1 y - v_1 \\ \dot{y} = v_2 \sin\phi - \omega_1 x \\ \dot{\phi} = \omega_2 - \omega_1. \end{cases} \tag{5}$$

The state space for the system (5) is taken to be

$$M \triangleq \Re^1 \times \Re^1 \times [0, 2\pi).$$

For any given $u \triangleq (v_1, \omega_1, v_2, \omega_2)$ such that $\omega_1 \neq 0, \omega_2 \neq 0$, the function

$$F(x,y,\phi;u) \triangleq (y - \tfrac{v_1}{\omega_1} + \tfrac{v_2}{\omega_2}\cos\phi)^2 + (x - \tfrac{v_2}{\omega_2}\sin\phi)^2$$

stays constant along the relative (with respect to A_1) trajectory of the aircraft A_2. In other words, $F(x, y, \phi; u)$ is a first integral of (5).

Definition 12 Let $F(x,y,\phi;u)$, where $u = (v_1, \omega_1, v_2, \omega_2)$, be a first integral of the system (5). A u_0 *F-slice* $S(A^-, A^+)$ *of the control system (5)* is the set of states for which the value of F under given controls $u_0 = (v_1^0, \omega_1^0, v_2^0, \omega_2^0)$ lies between some fixed values A^- and A^+, i.e.

$$S(A^-, A^+) \triangleq \{ (x,y,\phi);\ (x,y,\phi) \in M \text{ and } A^- < F(x,y,\phi;u)_{|u=u_0} < A^+\}.$$

□

Lemma 1 *(Caines & Lemch 1998)*

(i) For any $0 \leq A^- < A^+ < \infty$, $S(A^-, A^+)$ is nonempty, precompact, and path-connected.

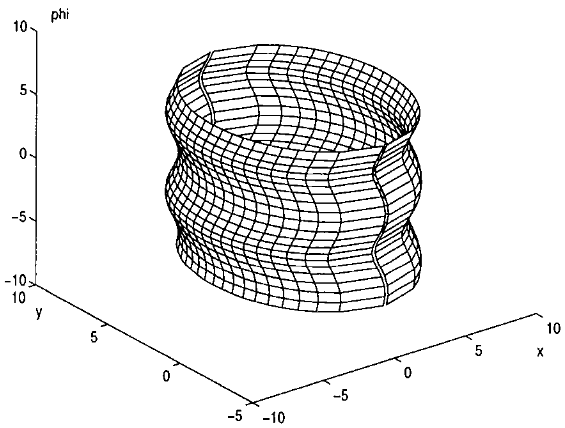

Figure 3: An F-slice of the system (5).

(ii) Consider the system (5) on some finite time interval $[T_0, T_1]$. Assume that the velocities $\bar{v}_1, \bar{\omega}_1$ of the aircraft A_1 are constant on $[T_0, T_1]$ and $\bar{\omega}_1 \neq 0$. Then each state $(x, y, \phi) \in M$ is a continuous fountain with respect to controls (v_2, ω_2).

(iii) The flow of the system (5) is measure preserving.

□

Applying the methodology described in Section 4, it can be shown that each u_0 F-slice (with $\omega_1 \neq 0, \omega_2 \neq 0$) constitutes a controllable subset of the system (5), where (v_1, ω_1) are treated as constant parameters and (v_2, ω_2) are treated as control inputs.

The air traffic system described above can be modelled as a hybrid system, in which each control location $q_i \in Q$ is characterized by the velocity (v_1^i, ω_1^i) of aircraft A_1. Then, as has been shown in the previous section, it is possible to construct a partition π_{q_i} in such a way that each block $X^{q_i} \in \pi_{q_i}$ is a u_0 F-slice of the system (5), for some $u_0 = (v_1^i, \omega_1^i, v_2^0, \omega_2^0)$. Hence $\Pi = \{\pi_{q_i};\ q_i \in Q\}$ is in-block controllable.

The partition machine with disturbances can be constructed as follows. The set of forbidden states F is taken to be the *relative protected zone*, which is the 5-mile radius cylinder around aircraft A_1. In (Tomlin, Pappas, & Sastry 1998), a methodology is developed in order to compute the predecessor $Pre(F)$; $Pre(F)$ is the set of initial states for which, regardless of the control input, there exists a disturbance input which would drive the system into the forbidden set. We declare a block $X^{q_i} \in \Pi$, for some $q_i \in Q$, to be *forbidden* or *safe* depending on whether the set of states $\{X^{q_i}\} \subset M$ satisfies, respectively,

(i) $Pre(F) \cap \{X^{q_i}\} \neq \emptyset$, or

(ii) $Pre(F) \cap \{X^{q_i}\} = \emptyset$.

For any disturbance input (v_1, ω_1) which is constant on $[T_0, T_1]$, each safe block $X \in \pi_q$ (where the discrete state $q \in Q$ corresponds to (v_1, ω_1)) is in-block controllable. Moreover, for any pair of safe blocks $X, Y \in \pi_q$ $(q \in Q)$, a disturbance rejecting control event U_X^Y is defined.

A control objective for aircraft A_2 may be formulated as follows: reach a certain point in the state space M, while (i) avoiding the protected zone F (or, for the high level controller, avoiding the set of forbidden blocks) (ii) minimizing a cost function (which could characterize, for example, fuel consumption, time, or any other resource). In other words, the objective is to balance the safety of each individual aircraft with the optimal utilization of resources.

Conclusion

Given a partition π, where each block $X \in \pi$ is a connected component of an energy slice, one can generate finer partitions using (i) $u = constant \neq 0$ energy slices, (ii) different integrals of motion, or (iii) controllable transversal manifolds, in particular *transversal foliations* (Broucke 1998). The resulting blocks may constitute neighbourhoods of system states of possible interest and hence form candidate targets for high level trajectory control. Even though, in general, the hybrid in-block controllability property is not preserved under the transversal decomposition of an energy slice block, the associated hybrid partition machine stays in the class of hybrid between-block controllable machines.

In conclusion, the notions of state aggregation described in this paper promise to facilitate the analysis and design of complex control systems; in addition to the traffic management example introduced above, we also mention in that context mechanical systems with many degrees of freedom and large space structures (Goh & Caughey 1985a; 1985b).

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References

Balluchi, A.; Benedetto, M. D.; Pinello, C.; Rossi, C.; and Sangiovanni-Vincentelli, A. 1998. Hybrid control for automotive engine management: The cut-off case. In *Proceedings of the first international workshop, Hybrid Systems: Computation and Control*, 13–32. Berkeley, CA: Springer-Verlag.

Branicky, M.; Borkar, V. S.; and Mitter, S. 1994. A unified framework for hybrid control. In *Proceedings of the 33rd IEEE Conference on Decision and Control*, 4228–4234.

Broucke, M. 1998. A geometric approach to bisimulation and approxamte verification of hybrid systems. In *Proceedings of the 37th IEEE Control Systems Society Conference on Decision and Control.*

Caines, P., and Lemch, E. 1998. On the global controllability of Hamiltonian and other nonlinear systems: fountain and recurrence. In *Proceedings of the 37th IEEE Control Systems Society Conference on Decision and Control*, 3575–3580.

Caines, P., and Lemch, E. 1999. *Hierarchical hybrid systems: geometry, controllability and applications to air traffic control.* To be presented at the Beijing IFAC World Congress, July, 1999.

Caines, P., and Wei, Y.-J. 1998. Hierarchical hybrid control systems: A lattice theoretic formulation. *IEEE Transactions on Automatic Control* 37(4):501–508.

Goh, C., and Caughey, T. 1985a. On the stability problem caused by finite actuator dynamics in the collocated control of large space structures. *Int. Journal of Control* 41(3):787–802.

Goh, C., and Caughey, T. 1985b. A quasi-linear vibration suppression technique for large space structures via stiffness modification. *Int. Journal of Control* 41(3):803–811.

Hammer, J. 1998. Handling disturbances in nonlinear control: the use of state feedback. *International Journal of Control* 70:761–787.

Jurdjevic, V. 1997. *Geometric Control Theory.* Cambridge University Press.

Kunita, H. 1979. On the controllability of nonlinear systems with applications to polynomial systems. *Applied Mathematics and Optimization* 5:89–99.

Lobry, C. 1974. Controllability of nonlinear systems on compact manifolds. *SIAM J. Control* 12:1–4.

Lygeros, J.; Godbole, D.; and Sastry, S. 1998. Verified hybrid controllers for automated vehicles. *IEEE Transactions on Automatic Control* 43(4):509–521.

Manikonda, V., and Krishnaprasad, P. 1997. Controllability of Lie-Poisson reduced dynamics. In *American Control Conference*, 2203–2207. Albuquerque, New Mexico.

Perry, T. 1997. In search of the future of air traffic control. *IEEE Spectrum,* August, 19–35.

Tittus, M., and Egardt, B. 1998. Control design for integrator hybrid systems. *IEEE Transactions on Automatic Control* 37(4):491–500.

Tomlin, C.; Pappas, G.; and Sastry, S. 1998. Conflict resolution for air traffic management: A study in multiagent hybrid systems. *IEEE Transactions on Automatic Control* 43(4):509–521.

Wei, Y.-J. 1995. *Logic Control: Markovian Fragments, Hierarchy and Hybrid Systems.* Ph.D. Dissertation, Department of Electrical and Computer Engineering, McGill University, Montreal, Canada.