Congruency of Beliefs

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Abstract

We present theoretical foundations and computational procedures of a theory for analysing decisions under risk when the available information is vague and imprecise. The impreciseness is expressed by a set of global distributions T over a space S, where the latter represents the classes of all probability and utility measures over a set of discrete outcomes. We show how local distributions, i.e. distributions over projections of S on various subspaces of S, can be derived from T and investigate in which extent user-asserted local distributions can be used for defining T. We also study invariants under local projections. The approach allows a decision maker to be as deliberately imprecise as she feels natural, as well as provides her with the means for expressing varying degrees of imprecision in the input sentences.

Keywords: Decision Analysis, Decision Theory, Utility Theory, Uncertain Reasoning

Background

The requirement of providing numerically precise data when analysing decision problems has often been considered unrealistic in real-life situations, and a number of models with representations allowing imprecise statements have been suggested. Some of them use standard probability theory while others contain some specialised formalism. Already in [Choquet, 1953/54] the concept of capacities was introduced, and later these ideas were studied in connection with probability theory [Huber, 1973, Huber and Strassen, 1973]. Logical approaches have also been used for providing methods for how to deal with sentences with upper and lower probabilities [Nilsson, 1986]. Belief states have been defined by interval-valued probability functions by means of classes of probability measures, and integrated in classical probability theory [Good, 1962, Smith, 1961]. [Dempster, 1967] investigated the properties of multi-valued mappings and defined upper and lower probabilities in terms of these. The results of Dempster were further developed into a non-Bayesian approach for quantifying subjective judgements [Shafer, 1976].[1] Fuzzy set theory is a widespread approach to relaxing the requirement of numerically precise data and providing a more realistic model of the vagueness in subjective estimates of values and probabilities, see, e.g., [Chen and Hwang, 1994, Lai and Hwang, 1994]. These approaches allow, among other features, the decision maker to model and evaluate a decision situation in vague linguistic terms and introduce various (often complicated) rules for aggregating this information. The measures defined are local and it is often hard to get an intuitive understanding of the global meaning of the various combinations of these. An important difference between fuzzy approaches and the approach presented in this paper is that the latter introduces global belief distributions with weak restrictions. We also show how to derive admissible classes of local distributions from sets of global distributions. This makes it possible to investigate the restrictions that have to be imposed on user-asserted local distributions depending on which information a decision maker has access to and provides. Furthermore, fuzzy approaches are restricted in the sense that they do not handle all qualitative aspects such as, e.g., comparisons between different components involved in many decision situations.

Quite general approaches for how to evaluate imprecise decision situations are investigated in [Gärdenfors and Sahlin, 1982, 1983, Levi, 1974, 1980]. The authors consider global distributions of beliefs, but restrict themselves to the probability case and, like the fuzzy models, interval representations. Another limitation is that they neither investigate the relation between global and local distributions nor introduce methods for determining the consistency of user-asserted sentences. The latter may in real-life decision situations be crucial, since the only information at hand in a decision situation may be local because most agents are not able to perceive their global beliefs over, for instance, a 53-dimensional probability base (which is

[1]As has been pointed out, the Dempster-Shafer representation seems to be unnecessarily strong with respect to interval representation [Weichselberger and Pöhlman, 1990].

actually a quite tricky task). The same criticism applies to [Hodges and Lehmann, 1952, Hurwicz, 1951, Wald, 1950]. The work by [Danielson and Ekenberg, 1997ab, Ekenberg, et al., 1996, 1997, Malmnäs, 1994] is restricted in the sense that no distributions over the intervals are taken into account.

The remaining parts of the paper describe how impreciseness can be modelled and discuss some general properties of global belief distributions. In particular, it is described how global belief distributions can be defined over a space and in what sense such distributions can define solution sets to a set of constraints, and how classes of admissible local belief distributions can be derived from projections of global distributions. Or conversely, what restrictions to be imposed on a subset K of local distributions given a set L of distributions so that $K \cup L$ defines a global belief distribution.

General invariants under local projections are defined, and specific instances are used, i.e. centroids and endpoints of intervals. Differents decision rules focusing on specific aspects can be studied by the use of corresponding invariants. Invariants are properties belonging to the global distribution which are meaningful to specify and to study for example in user-asserted local distributions.

Representation

The motivation behind the present work is to extend the expressibility when representing and evaluating vague and numerically imprecise information in decisions situations. To achieve a basic intuition of what will be presented below, consider a decision situation consisting of a set of n alternatives

$$\{\{c_{ij}\}_{j=1,\ldots,m_i}\}_{i=1,\ldots,n}$$

where each alternative is represented by a set of m_i consequences. We will refer to the latter as a consequence set. In such a decision situation, numerically imprecise sentences like "the probability of consequence c_{11} is greater than 40%" or comparative sentences like "consequence c_{11} is preferred to consequence c_{12}" occur. These sentences can be represented in a numerical format [Danielson and Ekenberg, 1997a]. Examples of vague sentences in that model are: "The consequence c_{ij} is probable" or "The event c_{ij} or c_{ik} is possible". Such sentences are represented by suitable intervals. Another kind of sentences are interval sentences of the form: "The probability of c_{ij} lies between the numbers a_k and b_k", which are translated to $p_{ij} \in [a_k, b_k]$. Finally, comparative sentences are of the form: "The probability of c_{ij} is greater than the probability of c_{kl} ". Such a sentence is translated into an inequality $p_{ij} \geq p_{kl}$. Each statement is thus represented by one or more constraints. The conjunction of constraints of the types above, together with $\sum_{j=1}^{m_i} p_{ij} = 1$ for each consequence set $\{c_{ij}\}_{j=1,\ldots,m_i}$ involved, is a probability base ($\mathcal{P}$). A value base ($\mathcal{V}$) consists of similar translations of vague and numerically imprecise value estimates. In a sense, a probability base can be interpreted as constraints defining the set of all possible probability measures. In the terms of [Gärdenfors and Sahlin, 1982], it defines a set of all epistemologically possible probability distributions.

Example 1: The solution set to the probability base $\{p_{11} + p_{12} + p_{13} = 1, p_{11} \geq 0, p_{11} \leq 0.6, p_{12} \geq 0.3, p_{12} \leq 0.5, p_{13} \geq 0.1, p_{13} \leq 0.5\}$ is a polytope. Each vector in the polytope corresponds to a probability distribution over the consequence set $\{c_{11}, c_{12}, c_{13}\}$. Thus, the polytope is a subspace of the space of all possible probability distributions over $\{c_{11}, c_{12}, c_{13}\}$ with respect to the probability base.[2]

However, a decision maker does not necessarily believe with the same intensity in all the epistemologically possible probability distributions E. To enable a refinement of the model to allow for a differentiation of distributions in this respect, a global distribution expressing various beliefs can be defined over the set E.

Example 2: We can use a function g representing[3] beliefs of the possible probability distributions with respect to the decision situation in example 1. Note that $g(x) > 0$ when x is an epistemologically possible probability distribution, i.e. is a vector in the polytope in Example 1.

Similar belief distributions can be defined over a set of value distributions, expressing epistemologically possible value distributions, where the latter is defined over consequence sets, for instance in terms of polytopes in the sense of [Danielson and Ekenberg, 1997a].

In the following subsections, we define and investigate some features of global distributions and how these are related to sets of linear constraints.

Consequence Sets and Bases

The basic entities in the kinds of decision situations we will consider are the sets of consequences involved. Over these sets, different functions can be defined, expressing for instance, classes of probability measures, belief functions in the Dempster–Shafer sense, fuzzy measures, or utility functions.

[2]Note that in general a set of probability distributions does not necessarily have to be convex. For instance, given the consequence set $\{c_{11}, c_{12}, c_{13}\}$, $(p_{11}, p_{12}, p_{13}) \in ([0, 0.2] \times [0.8, 1] \times [0, 0]) \cup ([0.3, 0.6] \times [0.1, 0.2] \times [0.2, 0.6])$ does define a set of probability distributions in the same way as above. The approaches in, for instance, [Danielson and Ekenberg, 1997, Ekenberg, et al., 1996, 1997, Levi, 1974, Malmnäs, 1994] assume convex sets and are for this reason of restricted use in cases where this is an adequate representation.

[3]To simplify the geometrical intuition, the function g can be defined over the natural projection of plane $p_{11} + p_{12} + p_{13} = 1$ on the p_{11}–p_{12} plane.

Definition 1 *Let Θ be a set of outcomes. A consequence set $C_i = \{c_{ij}\}_{j=1,\ldots,m_i}$, is a set of events such that $c_{ij} \cap c_{ik} = \emptyset$, for all $j \neq k$, and $\cup_{j=1}^{m_i} c_{ij} = \Theta$. A decision situation is a set, $\{\{c_{ij}\}_{j=1,\ldots,m_i}\}_{i=1,\ldots,n}$, where each $\{c_{ij}\}_{j=1,\ldots,m_i}$ is a consequence set.*

Definition 2 *Let a consequence set C be given. By a c-function over C, we mean a function $f : 2^C \to [0,1]$. By a generating mapping over C, we mean a set of c-functions over C, where each c-function f has the property $f(\emptyset) = 0$.*

Example 3: Let the consequence set $C = \{c_{11}, c_{12}, c_{13}\}$ be given. One c–function over C is $\{(\emptyset, 0.0), (\{c_{11}\}, 0.3), (\{c_{12}\}, 0.2), (\{c_{13}\}, 0.5), (\{c_{11}, c_{12}\}, 0.5), (\{c_{11}, c_{13}\}, 0.8), (\{c_{12}, c_{13}\}, 0.7), (\{c_{12}, c_{13}, c_{13}\}, 1.0)\}$. The images of all c–functions over C constitutes a 7–dimensional unity cube. The set of all c– functions over C is an example of a generating mapping over C.

Definition 3 *Let a consequence set*

$$C_i = \{c_{ij}\}_{j=1,\ldots,m_i}$$

be given. By the cell generating mapping over C_i, we mean the unity cube $[0,1]^{m_i}$. This space will be called a cell *for C_i. Below, such a cell will be denoted by $B = (b_{i1}, \ldots, b_{im_i})$ or, for notational convenience, $B = (b_1, \ldots, b_k)$. By a cell for c_{ij}, we mean the interval $[0,1]$. Such a cell will be denoted $B = (b_{ij})$ or $B = (b_i)$.*

Example 4: Let the consequence set $C = \{c_{11}, c_{12}, c_{13}\}$ be given. A cell for C is the space $[0,1] \times [0,1] \times [0,1]$.

Definition 4 *Let a decision situation*

$$D = \{\{c_{ij}\}_{j=1,\ldots,m_i}\}_{i=1,\ldots,n}$$

and a cell B_i for $\{c_{ij}\}_{j=1,\ldots,m_i}$ be given. By a cell *for D, we mean the space defined by $B_1 \times \ldots \times B_n$. Such a cell will be denoted $B = (B_1, \ldots, B_n)$.*

Example 5: Let a decision situation $D = \{\{c_{11}, c_{12}\}, \{c_{21}, c_{22}, c_{23}\}\}$ be given. A cell for D is the space $[0,1]^5$.

Global Belief Distributions

As was mentioned above, an agent does not necessarily believe with the same faith in all possible functions that the vectors in a cell define. For instance, when the agent considers a class of probability distributions, a reasonable requirement seems to be that the belief should be 0 in a vector where the mapping does not add up to one. To enable for a differentiation of functions in this respect, a global distribution expressing various beliefs can be defined over a cell.

Definition 5 *Let a cell $B = (b_1, \ldots, b_k)$ be given. By a global belief distribution over B, we mean a positive distribution*[4] *g defined on the cell B such that*

$$\int_B g(x)\, dV_B(x) = 1,$$

where V_B is some k–dimensional Lebesque measure on B. The set of all global belief distributions over B is denoted by $\mathrm{GBD}(B)$.

Example 6: Consider a uniform global belief distribution over a cell $B = (b_1, b_2)$. One interpretation of this is that we have no information about the consequence sets.

Example 7: The functions

$$f(x_1) = \max(0, \min(-100x_1 + 20, 100x_1))$$

and

$$h(x_2) = \max(0, \min(-\frac{100}{3}x_2 + \frac{80}{3}, \frac{200}{3}x_2 - \frac{100}{3}))$$

have graphs consisting of triangles with bases on the axes and area = 1. Then we get a global belief distribution $g(x_1, x_2) = f(x_1) \cdot h(x_2)$ over a cell $B = (b_1, b_2)$.

In the examples above we saw how to use global belief distributions to represent subsets of a cell. If we want to represent a subset which is of lower dimension than the cell itself we cannot use distributions that are upper bounded since a mass under such a distribution will be 0 while integrating with respect to some Lebesgue measure defined on the cell. Instead, we have to use, for example, the Dirac function $\delta_p(x)$ which has the property[5]

$$\int_B \delta_p(x) f(x)\, dx = f(p),$$

and especially, if $f(x) \equiv 1$ then

$$\int_B \delta_p(x)\, dx = 1$$

[4]A distribution on a set Ω is a linear functional defined on $C_0^\infty(\Omega)$ which is continuous with respect to a certain topology.

[5]For a detailed treatment, cf. for example [Friedlander, 1982].

so $\delta_p(x)$ is a global belief distribution according to the definition above. Here $f(x)$ is a measurable function. The distribution $\delta_p(x)$ is called *the Dirac distribution, or Delta function, with pole at the point* p.

Thus, we can use Dirac functions to represent pointwise global belief distributions.
In general, for every measurable subset E in a k-dimensional space B there is a distribution g_E with mass equal to 1 on E and 0 otherwise (cf. [Friedlander, 1982]).

Definition 6 *Let A be a subset of a cell B, and let $f \in \mathrm{GBD}(A)$. The natural extension $\tilde{f}_A(x)$ of f with respect to A is defined by*

$$\tilde{f}_A(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition 7 *Let A be a subset of B. A distribution $g_A \in \mathrm{GBD}(B)$, is called a characteristic distribution for A in B, if*

$$f(p) = \int_B \delta_p(x) \tilde{f}_A(x) g_A(x)\, dV_B(x),$$

for every $f \in \mathrm{GBD}(A)$, and for every point p in A.

From distribution theory follows that for every measurable subset A in a cell B, there exists a characteristic distribution for A in B. It also trivially follows that $\tilde{f}_A(x) \cdot g_A(x) \in \mathrm{GBD}(B)$ and equals 0 outside A.

Example 8: Let a cell $B = (b_1, b_2, b_3)$ be given. Let A denote the subset of B, where $x_1 + x_2 + x_3 = 1$, and let $f(x_1, x_2) = x_1 \cdot x_2$ be defined on A. Then $f \in \mathrm{GBD}(A)$ with respect to the 2-dimensional Lebesgue measure on A, but $\tilde{f}_A(x) \notin \mathrm{GBD}(B)$, because $\int_B \tilde{f}_A(x)\, dV_B(x) = 0$. However, $\int_B \tilde{f}_A(x) g_A(x)\, dV_B(x) = 1$, so $\tilde{f}_A(x) g_A(x) \in \mathrm{GBD}(B)$, and $\tilde{f}_A(x_1, x_2, x_3) g_A(x_1, x_2, x_3) = 0$, except when $x_1 + x_2 + x_3 = 1$. To put this informally; $\tilde{f}_A \cdot g_A$ represents the same proportional belief over B as f does over A.

Constraints

It should be noted that one property of a global belief distribution is that it in some sense defines the solution set to a set of constraints.

Definition 8 *Let a cell $B = (b_1, \ldots, b_k)$ be given. We will use the term constraints for the union of the following:*[6]

- *I-constraints are constraints on the form $a \geq x_i$ or $a \leq x_i$, where a is a real number in $[0, 1]$, and x_i is a variable.*
- *L-constraints are constraints on the form $\sum x_i = a$, where a is a real number in $[0, 1]$.*
- *C-constraints are constraints on the form $x_i \leq x_j + a$, where a is a real number in $[0, 1]$.*

[6]Naturally, the set of constraints can be more generally defined, but these three are sufficient for the purposes of this work.

Definition 9 *Let a cell $B = (b_1, \ldots, b_k)$ and a set C of constraints in $x_1, \ldots, x_k$ be given. The set of solution vectors to C constitutes the solution set for C, and will be denoted by $s(C)$. If there is a non-empty solution set for C, it is consistent. Otherwise C is inconsistent.*

Linear constraints can be used to model vague and numerically imprecise probability- and value statements. The introduction of global belief distributions over cells generalises the concept of probability- and value bases.

Definition 10 *Let a cell $B = (b_1, \ldots, b_k)$ and a distribution g over B be given. The support of g ($\mathrm{supp}\, g$) is the closure of the set $\{(x_1, \ldots, x_k) : g(x_1, \ldots, x_k) > 0\}$.*

Example 9: A value base V can be defined through a global belief distribution. Given a cell $V = (v_1, v_2)$ and a distribution g_V over V defined by $g_V(v_1, v_2) = 6 \cdot \max(v_1 - v_2, 0)$. Then $g_V \in \mathrm{GBD}(V)$, and $\mathrm{supp}\, g_V = \{(v_1, v_2) : 0 \leq v_i \leq 1 \,\&\, v_1 > v_2\}$.

Local Belief Distributions and Invariants

This section investigates relationships between global and local distributions and introduces measures for determining the consistency of user-asserted sentences. Such relationships are important, since the only information at hand in a decision situation may be local. We show how classes of admissible local belief distributions can be derived from projections of global distributions, and what restrictions to be imposed on a subset K of local distributions given a set L of distributions so that $K \cup L$ defines a global belief distribution.

Definition 11 *Let a cell $B = (b_1, \ldots, b_k)$ be given. By a local belief distribution over B, we mean a positive distribution f defined on the cell b_i such that*

$$\int_{b_i} f(x_i)\, dV_{b_i}(x_i) = 1,$$

where V_{b_i} is some Lebesque measure on b_i. The set of all local belief distributions over b_i is denoted by $\mathrm{LBD}(b_i)$.

Centroids

Local belief distributions over the axes of a cell B can be derived from a global belief distribution over B.

Definition 12 *Let a cell* $B = (b_1, \ldots, b_k)$ *and* $F \in \mathrm{GBD}(B)$ *be given. Let*

$$f_i(x_i) = \int_{B_i^-} F(x)\, dV_{B_i^-}(x)$$

where $B_i^- = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_k)$. *We say that* $f_i(x_i)$ *is derived from* F.[7]

In [Ekenberg and Thorbiörnson, 1997] we show how the use of centroids logarithmically reduces the computational complexity in the evaluations of a generalised expected utility. Intuitively, the centroid of a distribution is a point in space where some of the geometrical properties of the distribution can be regarded as concentrated.

Definition 13 *Let a cell* $B = (b_1, \ldots, b_k)$ *and* $g_B \in \mathrm{GBD}(B)$ *be given. The centroid of* g_B *is the point* $x_{g_B} = (\beta_1, \ldots, \beta_k)$ *in* B *whose i:th component is*

$$\beta_i = \int_B x_i \cdot g_B(x)\, dV_B(x).$$

Definition 14 *Let a cell* $B = (b_1, \ldots, b_k)$ *and* $f_{b_i} \in \mathrm{LBD}(b_i)$ *be given. The centroid of* f_{b_i} *is the point in* b_i *defined by*

$$x_{f_{b_i}} = \int_{b_i} x_i \cdot f_{b_i}(x_i)\, dV_{b_i}(x_i).$$

Centroids are invariant under projections on the local cells in the sense that the projection of the centroid on the local cell has the same coordinates as the centroid of the corresponding derived local belief distribution.

Lemma 1 shows that a product of local belief distributions has the same centroid as the distribution of a product.

Lemma 1 *Let a cell* $B = (b_1, \ldots, b_k)$ *and* $F \in \mathrm{GBD}(B)$ *be given. Let* $f_i(x_i)$ *be derived from* F. *Furthermore, let*

$$G(x_1, \ldots, x_k) = f_1(x_1) \cdot \ldots \cdot f_k(x_k).$$

Then

(i) $f_i(x_i) \in \mathrm{LBD}(b_i),\ i = 1, \ldots, k.$

(ii) $G \in \mathrm{GBD}(B)$[8]

(iii) $x_G = x_F$

(iv) If $x_G = (\alpha_1, \ldots, \alpha_k)$ *then* $\alpha_i = x_{f_{b_i}}$.

Proof:

(i): All we need to show is that $\int_{b_i} f_i(x_i)\, dx_i = 1$. According to the definition of f_i, we get

$$\int_{b_i} f_i(x_i)\, dx_i = \int_{b_i} \int_{B_i^-} F(x)\, dV_{B_i^-}(x)\, dx_i = \int_B F(x)\, dV_B(x) = 1.$$

(ii): $\int_B G(x)\, dV_B(x) = \int_B f_1(x_1) \cdot \ldots \cdot f(x_k)\, dx_1 \ldots dx_k = \left(\int_{b_1} f_1(x_1)\, dx_1\right) \cdot \ldots \cdot \left(\int_{b_k} f_k(x_k)\, dx_k\right) = 1 \cdot \ldots \cdot 1.$

(iii-iv): Let $x_G = (\alpha_1, \ldots, \alpha_k)$, and $x_F = (\beta_1, \ldots, \beta_k)$. We show that $\alpha_1 = x_{f_{b_1}} = \beta_1$. That $\alpha_i = x_{f_{b_i}} = \beta_i$ for $i = 2, \ldots, k$, follows by analogous calculations.

$\alpha_1 = x_{f_{b_1}}$, since

$$\begin{aligned} \alpha_1 &= \int_B x_1 \cdot G(x)\, dV_B(x) \\ &= \int_{b_1} \left(x_1 \cdot f_1(x_1) \cdot \int_{b_2} f_2(x_2)\, dx_2 \cdot \ldots \cdot \int_{b_k} f_k(x_k)\, dx_k \right) dx_1 \\ &= \int_{b_1} (x_1 \cdot f_1(x_1) \cdot 1 \cdot \ldots \cdot 1)\ dx_1 = x_{f_{b_1}} \end{aligned}$$

and $x_{f_{b_1}} = \beta_1$ since

$$\int_{b_1} x_1 \cdot f_1(x_1)\, dx_1 = \int_{b_1} x_1 \int_{B_1^-} F(x)\, dV_{B_1^-}(x)\, dx_1 = \int_B x_1 \cdot F(x)\, dV(x) = \beta_1,$$

because $f_i(x_i) = \int_{B_i^-} F(x)\, dV_{B_i^-}(x)$.

¿From the proof of Lemma 1 (ii) follows that a product of local distributions is a global distribution. Moreover, the following lemma strengthen this result by saying that if a global belief distribution G is a product of local belief distributions, then the factors are the derived local distributions from G.

Lemma 2 *Let a cell* $B = (b_1, \ldots, b_k)$ *be given. Let* $G(x_1, \ldots, x_k) = g_1(x_1) \cdot \ldots \cdot g_k(x_k)$, *where* $g_i \in \mathrm{LBD}(b_i)$. *Then* $g_i(b_i)$ *is derived from* G.

[7] In the following, we use x to denote the vector $(x_1, \ldots, x_k)$.

[8] In general, measure properties defined locally are not necessarily preserved globally, cf. [Thorbiörnson, 1996].

Proof:

$$\int_{B_i^-} g_1(x_1)\cdot\ldots\cdot g_k(x_k)\,dV_{B_i^-}(x)$$
$$= \int_B g_1(x_1)\cdot\ldots\cdot g(x_k)\,dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_k$$
$$= g_i(x_i)\cdot\left(\int_{b_1} g_1(x_1)\,dx_1\right)\cdot$$
$$\ldots\cdot\left(\int_{b_{i-1}} g_{b-1}(x_{i-1})\,dx_{i-1}\right)\cdot$$
$$\left(\int_{b_{i+1}} g_{b+1}(x_{i+1})\,dx_{i+1}\right)\cdot$$
$$\ldots\cdot\left(\int_{b_k} g_k(x_k)\,dx_k\right)$$
$$= g_i(x_i)\cdot\ldots\cdot 1$$

Relations Between Constraints and Belief Distributions

Of particular interest is to what extent local belief distributions can combine to a global belief distribution, so that the global distribution in some sense represents the local belief distributions as well as a set of constraints imposed on the decision situation.

Definition 15 *Let a cell $B = (b_1,\ldots,b_k)$ and a consistent set C of constraints in B be given. The global belief distribution F is called C-admissible iff*

x is a solution vector to C iff $x \in \operatorname{supp} F$.

Furthermore, a set of local belief distributions will be called I_C- admissible if the support of the distributions are congruent with the solution set to a set C of I-constraints.

Definition 16 *Let a cell $B = (b_1,\ldots,b_k)$ and a consistent set C of I-constraints in B be given. A set $L = \{f_i(x_i) \in \mathrm{LBD}\,(b_i)\}_{i=1,\ldots,k}$ of local belief distributions is called I_C-admissible iff*

$(x_1,\ldots,x_k)$ is a solution vector to C iff $x_i \in \operatorname{supp} f_i$ for all $i = 1,\ldots,k$.

Usually a decision maker has access only to local information and a set of relations between different parameters and, consequently, has no explicit idea about the global distribution. In many cases, it may be that the only accessible relations between the local distributions are in terms of constraints. When only I-constraints are considered, this set of distributions can be used to define a global belief which support is positive only on the polytope defined by the constraints.

Theorem 1 *Let a cell $B = (b_1,\ldots,b_k)$ be given. Let C be a consistent set of I-constraints such that $s(C) \subseteq B$, and let a set $\{g_i(x_i)\}_{i=1,\ldots,k}$ such that $g_i \in \mathrm{LBD}\,(b_i)$ be given. Then there exists $F \in \mathrm{GBD}\,(B)$, such that F is C-admissible iff $\{g_i(x_i)\}_{i=1,\ldots,k}$ is I_C-admissible.*

Proof: Define $F = g_1\cdot\ldots\cdot g_k$. Then from Lemma 3 follows that $f_i = g_i$, where f_i is derived from F. From Lemma 1 follows that $F \in \mathrm{GBD}\,(B)$. Furthermore, F is I_C-admissible iff

$$x \in s(C) \iff x \in \operatorname{supp} F.$$

However, $F = 0$ iff $g_i = 0$ for some i, i.e. $(x_1,\ldots,x_k) \notin \operatorname{supp} F$ iff $x_i \notin \operatorname{supp} g_i$ for some $x_i \in (x_1,\ldots,x_k)$. This is equivalent to that the set $\{g_i(b_i)\}_{i=1,\ldots,k}$ is I_C-admissible.

Congruency

If the decision maker is able to define a set of local belief distributions and a set of L- and C-constraints describing the decision problem, these must be congruent in a certain respect. Given a set of constraints, a decision maker is restricted concerning which combinations of local belief distributions that are possible to impose, if she wants to be consistent in a reasonable sense. This is expressed by the following definition.

Definition 17 *Let a cell $B = (b_1,\ldots,b_k)$ and a consistent set C of constraints in B be given. A set $L = \{f_i(x_i) \in \mathrm{LBD}\,(b_i)\}_{i=1,\ldots,k}$ is called C- admissible iff the vector $(x_{f_1},\ldots,x_{f_k})$ is a solution vector to C, where x_{f_i} denotes the centroid of f_i.*

Theorem 2 *Let a cell $B = (b_1,\ldots,b_k)$ and a consistent set C of constraints, such that $s(C) \subseteq B$ be given. Let G be a C-admissible global distribution and let $g_i(b_i), i = 1,\ldots k$, be derived from G. Then $\{g_i(x_i)\}_{i=1,\ldots,k}$ is a C-admissible set of local belief distributions. Furthermore, if $F = g_1\cdot\ldots\cdot g_k$, then $x_F = x_G$.*

Proof: The second part of the theorem follows immediately from Lemma 1, i.e. $x_F = (x_{f_1},\ldots,x_{f_k}) = (x_{g_1},\ldots,x_{g_k}) = x_G$. The first part of the theorem follows from Lemma 1 and the observation, from standard convexity theory, that the solution set to a set of linear constraints is convex.

Theorem 1 and 2 imply that if a decision maker defines a set of local belief distributions describing a problem, and if these are admissible w.r.t. the constraints involved, a global belief distribution can be determined. This distribution has the property of having the same centroid (and the same support relative to the local belief distributions) as any global belief distribution from which the user-asserted local belief distributions can be derived.

Defining Admissible Belief Distributions

An important issue in this context is to simplify the interactive processes for decision makers when the user-defined belief distributions are not admissible with respect to the definitions above. One prima facie solution is to form the vector of the centroids of the user-defined distributions and choose the nearest vector that consists of centroids of an admissible set of local belief distributions. Since we are primarily interested of the centroids and not of the distributions themselves, we do not have to present the possible candidates explicitly. The concept of nearest vector could, for instance, be defined in terms of the Euclidean distance. Various kinds of sensitivity analyses could also be introduced by moving around this vector in the solution set to the constraints.

If a decision maker more explicitly wants to investigate the effects of altering the user-defined distributions, another suggestion would be to define various complete subsets of them. When there are linear dependencies involved (as an effect of a set of L-constraints), then the sets of all local belief distributions are, in a sense, determined by a subset of such a set. We will say that this subset is complete.

Definition 18 *Let a cell* $B = (b_1, \ldots, b_k)$ *be given. Let* C *be a set of* I *and* L*-constraints in* B*, and let* $L = \{f_i(x_i)\}_{i=1,\ldots,k}$ *where* $f_i(x_i) \in \text{LBD}(b_i)$*. A subset* $\{f_i(x_i)\}_{i=t_1,\ldots,t_s}$ *of* L *is* B*-complete if the dimension of* $(b_{t_1}, \ldots, b_{t_s})$ *is equal to the dimension of the solution set to* C*. A* B*-complete subset* $\{f_i(x_i)\}_{i=t_1,\ldots,t_s} \subseteq L$ *is* C^B_{COMP}*- admissible if the vector* $(\alpha_1, \ldots, \alpha_s)$ *lies in the natural projection of* $s(C)$ *on* $(b_{t_1}, \ldots, b_{t_s})$*, where* $\alpha_i = x_{f_{t_i}}$ *is the centroid of* f_{t_i}*.*

The following theorem says that if $\{f_i(x_i)\}_{i=t_1,\ldots,t_s}$ is a C^B_{COMP}-admissible set, then the admissible local distributions over $(b_1, \ldots, b_k)$ outside $(b_{t_1}, \ldots, b_{t_s})$ are determined.

Theorem 3 *Let a cell* $B = (b_1, \ldots, b_k)$ *be given. Let a set* C *of* I *and* L*-constraints in* B*, such that* $s(C) \subseteq B$*, and let a set* $M = \{f_i(x_i)\}_{i=t_1,\ldots,t_s}$*, such that* $f_i \in \text{LBD}(b_i)$*, be given. Assume that* M *is* B*-complete. Then the* C*-admissible set of distributions* $L = \{f_i(x_i)\}_{i=1,\ldots,k}$ *is uniquely defined when* $M \in L$*. Furthermore,* $x_F \in s(C)$ *iff all* B*-complete subsets of* L *are* C^B_{COMP}*-admissible.*

Proof: The second part of the theorem follows immediately from Theorem 2. Assume that $G(x_1, \ldots, x_k) \in GBD(B)$ and that L is a set of derived local distributions from G. Since a set of n linearly independent L-constraints determines $k - n$ variables, we can define a function $F(x_{r_1}, \ldots, x_{r_q}) = G(x_1, \ldots, x_k)$, where $\{r_1, \ldots, r_q\} = \{1, \ldots, k\} - \{t_1, \ldots, t_s\}$. Now $\{f_i(x_i)\}_{i=r_1,\ldots,r_q}$ are derived from G iff they are derived from F.

When a decision maker asserts local belief distributions, they do not always fulfill the admissibility described above. One option for the decision maker in such cases is to select a C^B_{COMP}-admissible subset of local distributions and then calculate the remaining ones.

Concluding Remarks

We have presented theoretical foundations and computational procedures of a theory for analysing decision situations including probability- and value estimates, when the available information is indeterminate. The approach allows a decision maker to be as deliberately imprecise as she feels is natural as well as provides her with the means for expressing varying degrees of imprecision in the input sentences. The main idea is that impreciseness is expressed by global belief distributions, expressing relative beliefs in different values. However, in many cases the only accessible information might be in terms of local beliefs and sets of relations. Therefore, we also investigate how local distributions, i.e. belief distributions over various subspaces of the solution sets to the probability-, and utility bases, can combine to global belief distributions that, in some reasonable sense, correspond to the local information as well as to the relations. Furthermore, we investigate which properties user-asserted local belief distributions should have to be congruent with the constraints involved, and how new local distributions can be derived from complete sets of local belief distributions. The model presented herein is based on positive measures, and another line of research is to base the theory to the more general concept of capacities, cf. [Thorbiörnson, 1998].

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