Theory-driven historical discovery: Boole's abstract formalization of Logic

Luis de Ledesma, Aurora Pérez, Daniel Borrajo and Luis M. Laita
Facultad de Informática, Universidad Politécnica de Madrid
28660 Boadilla del Monte, Madrid, Spain
{ledesma,aurora,dborrajo,laita}@fi.upm.es

Introduction

There is a trend in the field of Machine Discovery that has undertaken the task of approaching Scientific Discovery from a computational viewpoint. Such an approach is a very promising, and in some way fascinating, field of research in Artificial Intelligence. Its ultimate goal is to recognize and learn patterns of previous discovery, which can improve one's chances for future discovery significantly (Oliver 1991). Most studies have dealt with empirical laws of physics and chemistry but, so far, almost no theory driven discovery has been contemplated. The present paper deals with a computational approach to the real history of the genesis of a mathematical discovery: an abstract-algebraic method called "the method of separation of symbols", and its role in the creation of George Boole's logic. We have studied the different historical factors that influenced this creation. The real history of the discovery under consideration has suggested a computational model that provides the basis for a theory of this kind of discovery.

The paper presents a sketch of the history of Boole's discovery as well as the influence of Duncan F. Gregory on it, and describes the system BOOLE2. This program embodies Boole's method of discovering, discovers Logic and Geometry as parts of Algebra, and is also ready to be used on a variety of sciences.

The actual historic discovery

Preliminaries

Gregory and Boole were able interpreters and developers of a mathematical methodological tradition begun in France in the second half of the 18th century and continued in Britain during the first half of the 19th century. This tradition involved the methodological idea of separating symbols of operations from their subjects of application, and operating with the former as with algebraic entities. The method was called *the method of separation of symbols* and it was at the heart of Boole's discovery and development of *Logic* as an algebraic discipline.

Even though Gregory's contribution to the method itself is usually recognized as most substantial, it should be emphasized that only Boole had the clearness and boldness needed to try it on an extra mathematical discipline: *Logic*. The historical and detailed explanation of this process has been a substantial part of our research, and it will be briefly summarized in this first part in order to introduce a computer system that works according to the way we conceive Boole's discovery processes.

The historical development of the method has been studied in (Koppelman 1971; Knobloch 1981; Panteki 1991) as part of the history of Symbolic Calculi. More detailed accounts on Boole can be found in (Laita 1977; 1979; 1980; de Ledesma & Laita 1989).

Gregory's contribution to the method of separation of symbols

Gregory applies the method of separation of symbols to *Geometry* (Gregory 1839), giving in this way an indirect description of how the method works. In *Geometry*, the operations to be symbolized are related to the ideas of magnitude and direction, and the operation from which magnitude can be defined is, according to Gregory, transference in one direction. Gregory assumes the symbols $a, b, c,...$, to represent transference in one direction. The result of applying a to a point (.) is a segment, applying b to $a(.)$, written $b(a)$, is a parallelogram obtained by transference of all points of the first segment; similarly $c(b(a))$ will result in a parallelepiped. He says that there is no interpretation for more than three applications of the symbol of transference, the result of which may be considered as an "impossible" geometrical operation, in the same sense that "$\sqrt{-1}$ is an impossible arithmetical one".

In (Gregory 1839, p.3) Gregory proceeds to show that the combination of transferences is commutative, $a(b) = b(a)$, and distributive with respect to a sum operation $a + b$ which represents the translation of a point in the same direction through distances a and b, $c(a + b) = c(a) + c(b)$. Geometrical concepts are then represented by arithmetical symbols. In the eye of Symbolic Algebra, as arithmetical symbols of the sum and product and geometrical concepts of iteration of transferences follow the same laws of combination, they are identical (that is, belong to the same family). So, whatever can be proved in arithmetic in dependence of only those laws alone, is valid for the geometry of transference.

Boole and the method of separation of symbols

Boole's awareness of the advantages of facing mathematics as a calculus of symbols began very early in his life. In 1835, when he was twenty years old, he gave an address on Newton which seemed to refer to Boole's own philosophy of mathematics as much as to Newton's. The ideas on the necessity of a systematic use of symbolism as well as the possibility of separating the use of symbols from their interpretation were already there.

In 1839, Boole went for a trip to Cambridge and met the mathematician Duncan F. Gregory, who recently had founded the *Cambridge Mathematical Journal.* Boole submitted several papers for publication and the latter saw with astonishment that Boole had developed by himself -influenced only by his readings of Lagrange and other French mathematicians- an approach to mathematics very similar to the one developed at Cambridge. Reading, for instance, the paper (Boole 1841), is a fascinating experience on the application of an abstraction of the analytical change-of-variables technique to the symbolic transformation of a curve's equation.

The examination of some of Boole's subsequent papers shows that it was by this time when Gregory informed Boole of the particular traits of the method of separation of symbols. Boole recognizes that Gregory was his inspirer, but he went much further than Gregory. He applied the method to some truly difficult problems in the differential calculus and the calculus of finite differences. And, what is most important, Boole made this bold guess: symbolic calculus, and the method of separation of symbols in particular, could be applied outside mathematics, particularly to Logic. The reason that made Boole return his attention to Logic was a controversy about a logical subject between the philosopher William Hamilton and the mathematician Augustus De Morgan. He had been interested in Logic much longer before: "... I was induced by the interest which it (the controversy) inspired, to resume the almost forgotten thread of former inquires", (Boole 1847, p.1). And it is of utmost importance to see how, this second time, Boole founded Logic on the method of separation of symbols. As a matter of fact, the first principles of *Logic* as they appear at the beginning of the book *The Mathematical Analysis of Logic* (Boole 1847) are an almost direct transcription of laws suggested by Gregory in his version of the method of separation of symbols. Boole writes at the beginning of the mentioned book:

> Further, let us conceive a class of symbols x, y, z, possessed of the following character. The symbol x, operating upon any subject comprehending individuals or classes, shall be supposed to select from it all individuals of the class X which it contains... When no subject is expressed, we shall suppose 1 (The Universe) to be the subject understood, so that we shall have $x = x(1)$. The result of an act of election is independent of the grouping or classification of the subject... We may express this law mathematically by the equation $x(u+v) = xu + xv$. It is indifferent in what order two successive acts of election are performed. The symbolic expression of this law is $xy = yx$. The result of an act of election performed twice or any number of times in succession is the result of the same act performed once... supposing the same operation to be n times performed, we have $x^n = x$. (These) laws are sufficient for the basis of a calculus... The third law $x^n = x$ we shall denominate the index law. It is peculiar to elective symbols and will be found of great importance .. (Boole 1847, pp.15-18).

The last sentence in this quotation states how *Logic* turns out to be a calculus governed by the same laws as some of those in the method of separation of symbols. This is not just a fancy abstract statement. Actually, the whole book (Boole 1847) is a complete development of an Algebraic Logic on the basis of precisely those three laws.

Boole's discovery consisted of finding that *Logic* was among the sciences worth of symbolization. This means that its basic operations follow the same laws of combination of symbols as some of the families in Symbolic Algebra. From this point of view, Gregory's finding on *Geometry* can be considered as just another instance of the same process.

Description of BOOLE2

According to the extensive research that we have done about the life and work of G. Boole, we built first a program, BOOLE1, that, following Boole's reasoning process to the Mathematical Analysis of Logic, reached the same conclusions as he did (de Ledesma *et al.* 1993). This reasoning process tried, and succeeded, in applying the *Method of Separation of Symbols* to *Logic.* Since it was our firm belief that -in Boole's view- the process of finding whether a science is symbolizable or not (whether the method of separation of symbols is applicable to it or not), is always the same, we undertook the task of generalizing BOOLE1's heuristics to make it applicable to other sciences as well. Duncan F. Gregory's Geometry was chosen as a first instance. Gregory was not selected as the active agent of a discovery process to be formalized by a computer program (our purpose was to reach the conclusion of his Geometry being symbolizable by an emulation of G. Boole's reasoning), but as the author of a branch of Geometry particularly apt for our intention.

The formalization of the just mentioned ideas and aims resulted, then, in a program, BOOLE2, that using a uniform way of representing and reasoning is able to discover that both *Logic* and *Geometry* are symbolizable (de Ledesma *et al.* 1994). In fact, its representation and reasoning tools are intended to handle any science as a candidate for symbolization, provided its basic contents can be turned accessible to the system. In this way, the so called generic part of the system may be viewed as a module that has information about the method of separation of symbols, explicitly including the knowledge that is common to all sciences, i.e., a scheme designed to receive each specific science as an input: operations, combinations and the possible laws thereof. This enlarges the applicability of the system to different environments, besides *Logic*. So far, BOOLE2 has been provided with understanding on the way Gregory conceived *Geometry*. In this manner, it considers *Geometry* as a candidate for symbolization and reaches the laws of combination of the basic operation, transference, as stated by Gregory in (Gregory 1839), plus the conclusion of such *Geometry* being symbolizable.

The main difference between BOOLE2 and other systems dealing with the process of scientific discovery such as BACON, GLAUBER, STAHL and DALTON, (Langley *et al.* 1987), AM and EURISKO (Lenat 1982) or ARE (Shen 1990), is that the former is exclusively guided by the theory, instead of experimentation, i.e., it is not a data driven discovery system. First, and having in mind that "data driven" refers to the experimental character of the input, BOOLE2's input is an abstract representation of the science to be considered. On the other hand, a substantial part of its heuristics stems from theory and it should be emphasized that, in this case, theory means the very precise symbolic way by which Boole handled the method of separation of symbols. The method is a part of Mathematics and, therefore, demands an abstract way of reasoning. BOOLE2's starting point is the knowledge and goals previous to the discovery of a science being symbolizable or not, and it ends stating whether the method of separation of symbols is applicable to it or not. More specifically, it starts from the state of mind of G. Boole before "The Mathematical Analysis of Logic" and it reaches the conclusions about either Logic or Geometry by an emulation of Boole's own reasoning process.

The above paragraph makes clear which are the input and output for the system: the input is a description of the science we want BOOLE2 to consider (its operations, combinations and so on); the output is a record of its algebraic properties plus the conclusion of such science being symbolizable or not. Let us examine a first sketch of the process that leads to such an output. G. Boole always seemed to perfectly know his objectives and to have the ability to separate, in almost every instance, the right path to them from the infinity of all other conceivable wrong, useless or unduly long paths. This is the spirit that has been given to the BOOLE systems. Their heuristics are just the insightful reasoning method of Boole.

The process of execution of BOOLE2 may be described as follows. The initial goal is to incorporate a science to the set of symbolizable ones. Since this goal is not true yet, the system tries to achieve it by checking first whether the science deserves to be symbolized (a new goal: "worth of symbolization"). In this way, the top level goal needs a number of subgoals to be true, that, in turn, will need other subgoals and so on. This backward process is based on the above remarks on George Boole's reasoning. The execution ends when the top level goal is attained. The achievement of each goal simulates one step of the discovery process. Figure 1 shows the top level part of the subgoaling structure of BOOLE2's reasoning.

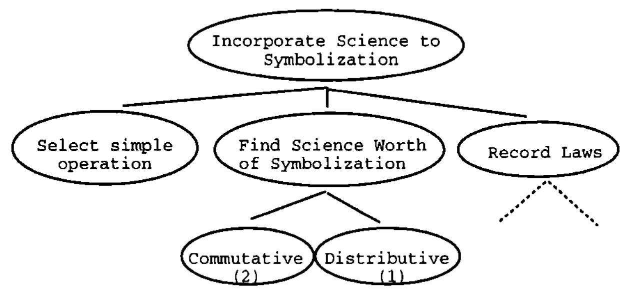

Figure 1: Top level part of the subgoaling structure generated by BOOLE2 when reasoning

The program has been formalized as a production system, with its three classical components: working memory, set of rules, and control. Its knowledge is divided in two: a generic part, and a specific one. The generic part is a set of rules and frame descriptions, common to any science, while the specific part is composed of a set of rules and frame descriptions for each particular science. This is precisely one of the strong points of the system, since it allows to create a *discoverer* for each science given a set of instantiated operations on that science. Figure 2 shows the architecture of the system.

The set of initial facts of the working memory comprises the set of all facts in *Logic* and *Geometry* prior to their consideration as symbolic disciplines. The intermediate and final sets of facts represent the different states of mind of the human scientist on his or her reasoning process. The set of rules describes the individual steps of the general process of finding whether a particular science can be symbolized or not, and how to perform simple operations on the respective sciences. The control mechanism guides BOOLE2 towards the same kind of reasoning that G. Boole used.

Remark on vocabulary.

The term operation is used in this context in a different sense than the usual one. This is due to historical reasons and reflects G. Boole's conception of sciences. An operation means something close to the modern

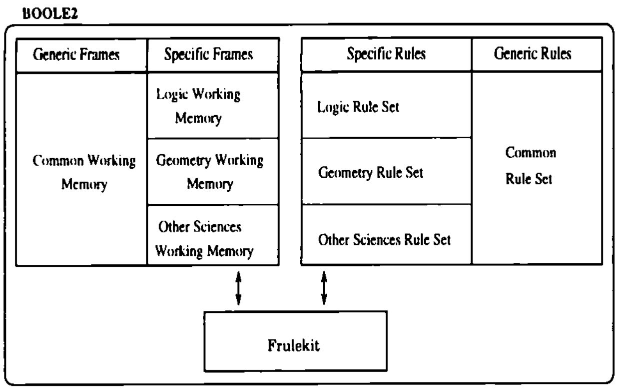

Figure 2: Architecture of BOOLE2

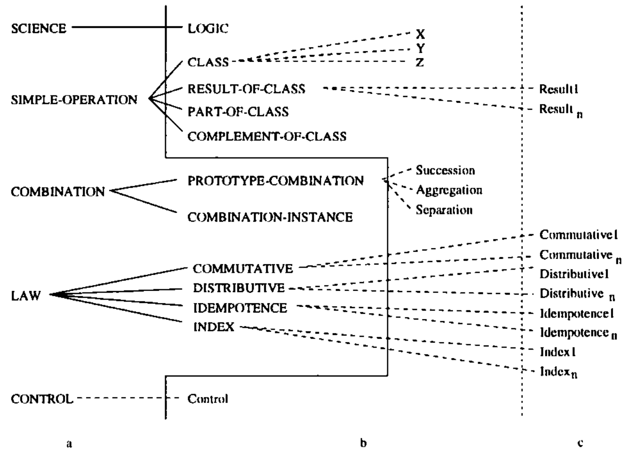

Figure 3: Hierarchy of frames on BOOLE2 for *Logic*

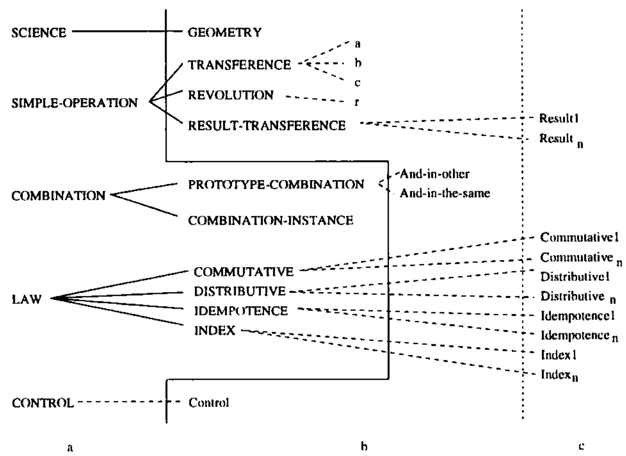

Figure 4: Hierarchy of frames on BOOLE2 for *Geometry*

idea of operator. For instance, differentiation, is an operation of Calculus; taking the trace of a point moving through a segment is an operation of *Geometry* ("transference"); and selecting a class of individuals from a universe is an operation of *Logic* ("class"). Notice that the effect of the operations in the three previous examples are the well known concepts of derivative, segment and predicate (identified with a set of individuals).

On the other hand, we use the term combination (of operations) much in the sense of today's algebraic operation. For instance, two successive acts of selecting individuals from class x, first, and, then, from class y ("class succession") yield the class of the individuals which are both x and y; and two translations of a point through different directions ("transference and-in-other") yield a parallelogram.

The Working Memory

The state of the working memory at a given moment of the execution of the production system represents a corresponding mental state of the discoverer in his/her reasoning process. In BOOLE2, the working memory is composed of a set of frame descriptions, together with a set of instances of those frames. Figures 3 and 4 show the hierarchy of frames.

The working memory is divided in three parts: static science-independent, static science-dependent, and dynamic. The static science-independent part is composed of the hierarchy of frames that is common to all sciences just before the process starts, i.e., the generic frames. This is shown in figures 3 and 4 by the set of frames that are on section a. The static science-dependent part is the set of specific frames and instances, created for each science in the initial state. This is shown on the figures by the set of frames and instances that are on section b. Finally, the dynamic part is composed of the specific instances that are created at run time, represented on section c.

As an example of frame and instance descriptions, figure 5 shows a science-independent frame and a science-dependent instance. Namely, the frame *science*, which has the description of the science, and the instance *succession*, which has the information on a combination of simple operations of the science *Logic*. The frame slots are self-explanatory. Nevertheless, some of the instance slots require a better understanding of how BOOLE2 handles the combinations of simple-operations. When BOOLE2 decides, through the rules, that a combination should be applied on two simple-operations, it creates a notation for each simple-operation. The notation is computed using the function value of the slot *notation-generator-1* of the corresponding combination.

A comment should be made on how *Logic* has been represented as an input to BOOLE2 (similar comments can obviously be given for *Geometry*, but we are referring to just an example). Any science representation must first state which is the operation, or operations, to be considered as simple. The simple operation for *Logic* is the class. According to Boole's ideas, each class, named x, represents both the act of selection of all individuals X included in that class, as well as

```
;;; Frame that describes a general science
(literalize science (:cache :*ALL*)
    :name nil
    :simple-operations nil
    :combinations nil
    :laws nil
    :worth-symbolization-p nil)

;;; Instance of the frame prototype-combination
;;; named succession. It describes how to perform
;;; a succession on two simple-operations on the
;;; science Logic.
($make 'prototype-combination
   :name 'succession
   :meaning 'succession
   :science 'logic
   :notation-generator-1 #'succession-ng
   :result-generator #'succession-rg)
```

Figure 5: A frame and an instance in BOOLE2

the result of that act (the set formed by, exactly, all individuals X). One of the notations used for, both, representing and handling x is the list (ALL THAT ARE X). Why such an "essentialist" representation? It is intended also to reflect Boole's abstract way of reasoning. It is possible to represent aggregation of classes as set union and infer by a short induction the same results obtained by BOOLE2, but that would be unloyal to the sort of linguistic way in which G. Boole handled this combination of operations (the aggregation of two classes). On the other hand, this notation, as well as the others that have been chosen, has shown to be particularly apt for a transparent representation, and also for performing the different combinations.

Rule set

The rule set is divided in two subsets: discovery rules, and recording rules. The discovery rule set is composed of rules that reason along the set of processes that can be applied to the elements of a science, and can be divided into the following sets:

- **General rules**. They control the abstract reasoning for considering a science worth symbolizable.
- **Law rules**. They represent the common laws for the sciences, such as *commutative, distributive* and *idempotence*.
- **Combination rules**. These rules perform the default combinations over the operations of the sciences. In particular, for the science *Logic*, the combinations are *succession*, and *aggregation*, and for the science *geometry* are *and-in-the-same-dimension* and *and-in-other-dimension*.
- **Forced combination rules**. These rules act as heuristics of discovery in the sense that they provide a way of combining operations when certain conditions hold. As an example, the forced-succession rule is a heuristic that says that the system should perform the combination of two operations *o2* and *o1*, if it has already performed the combination of *o1* and *o2*; this will help to fire the rule relative to the commutative law if the results of both combinations are the same.
- **Continuation rules**. These are rules that control the way in which rules are matched. They provide a control above the one supplied by the agenda mechanism. In particular, they do not allow the same type of combination to be applied twice in a row whenever another rule can be fired.

The second kind of rules, the recording rules, are rules that collect all laws that are satisfied in a certain science.

Control mechanism

The control of the execution is done through an agenda mechanism. Rules are organized in priority levels, usually with more abstract rules on higher levels of priority, and more detailed rules in lower levels of priority. Figure 6 shows the different levels with the type of rules on each level.

Recording rules
General rules
Law rules
Forced combinations rules
Combination rules
Continuation rules

Figure 6: High level description of BOOLE2's agenda.

The agenda follows a priority policy, by which the rules on each level are matched to see if anyone can fire.[1] If no rule of a certain level fires, the control goes to the next level. As soon as a rule of a certain level fires, the control goes to the highest level, in order to see if any of the high level rules can fire.

In order to suplement the agenda with more complex ways of control, BOOLE2 has the forced combination rules and the continuation rules. The first type of rules plays the role of heuristics based on the state of the search. They allow to control how to apply the discovery rules in a similar way to the one used by G. Boole. The second type of auxiliary control rules allow to explore the search space uniformely, not repeating the same basic step (combination) twice without trying out different combinations. These two types of heuristics are useful for guiding the process, and relating it to the real discoveries.

Conclusions

The main contributions of the research, are:

[1]Frulekit uses a Rete net for efficiently matching the rules.

- We have carried out an accurate and detailed study of the work of a human scientist, George Boole, in his particular view of the *Method of Separation of Symbols*, its scope and actual applicability to a science, *Logic*, which was not considered, by that time, as a mathematical discipline.
- The study gave us the guidelines to build a computational model that behaves like the scientist G. Boole in a given reasoning process, such as considering a science symbolizable. The resulting program, BOOLE2, behaves like a re-discoverer, in the sense that it reproduces real scientific discoveries. The execution of BOOLE2 with *Logic* as an input proves that the system is able to go all the way from the starting point to the conclusions of Boole himself. This shows that the knowledge given to the program by the authors of this paper is sufficient for explaining the way George Boole got to his discovery and, therefore, that BOOLE2 itself is an approximation to a theory of this scientific discovery.
- The results showed us that the process is general enough to be translated to other sciences, by just entering the initial description of the science to be considered. This has been the case for Gregory's *Geometry*.

We would like to emphasize that, even though the system does not mimic a large number of case studies, it captures an interesting level of detail for those that are actually studied. Just because this sound ground of detailed conceptualization, it can be considered as potentially able to handle other cases as well. The separation-of-symbols heuristic has been consistenly used from 19th century on with the label of "formalization" or "abstraction". Whenever a portion of knowledge is symbolically represented and its symbols ("separated" from the particular meaning they have in their domain) are found to behave in the same way as those of, say, Logic, Algebra, Graph theory, etc., a separation-of-symbols process has been carried out. It is in this sense that *any science* can be considered as a candidate-input for BOOLE2. Due to their low level of formalization, Boole's contemporary sciences are more fitted than others to the spirit of the present study. De Morgan's Logic is an excellent candidate for both its proximity in terms of personal relationship to George Boole and, at the same time, its almost null algebraic character. But any other discipline might be considered as well. There would always be an output, "it is, or it is not, symbolizable", provided we had devised an appropiate representation of such discipline, which is by no means a trivial task.

Acknowledgments
We owe thanks to Prof. Herbert A. Simon, for encouraging us in the past to undertake the task of creating the systems BOOLE and for many subsequent and extremely fruitful discussions prior to this paper, and to Prof. Raul E. Valdés-Pérez whose helpful comments have contributed to improve our work.

References

Boole, G. 1841. Researches on the theory of analytical transformations. *Cambridge Mathematical Journal* II:64–73.

Boole, G. 1847. *The mathematical analysis of logic, being an essay towards a calculus of deductive reasoning.* Macmillan, Barclay, and Macmillan.

de Ledesma, L., and Laita, L. M. 1989. George Boole. From differential equations to mathematical logic. In *Actas de la Reunión Matemática en honor de A.Dou,* 341–352. Madrid: Publicaciones de la Universidad Complutense.

de Ledesma, L.; Laita, L. M.; Pérez, A.; and Borrajo, D. 1993. Descubrimiento científico e inteligencia artificial. *Real Academia de Ciencias Exactas Físicas y Naturales* 115–132. Proceedings of the Segundo Curso de conferencias sobre Inteligencia Artificial.

de Ledesma, L.; Pérez, A.; Borrajo, D.; and Laita, L. M. 1994. La lógica de Boole como consecuencia del método de separación de símbolos de Gregory-Boole. Estudio histórico y su emulación por computador. In *Contra los titanes de la rutina. Contre les titans de la routine. Proceedings of the Congreso Hispano-Francés sobre la Historia y la Filosofía de la matemática.* Comunidad de Madrid/CSIC. 287–300.

Gregory, D. F. 1839. On the elementary principles of the application of algebraic symbols to geometry. *The Cambridge Mathematical Journal* 2:1–9.

Knobloch, E. 1981. *Mathematical perspectives, Essays on Mathematics and its historical development.* New York: J.M. Dauben. chapter Symbolik und Formalismus im mathematischen Denken des 19. und beginnenden 20. Jahrhunderts, 208–216.

Koppelman, E. 1971. The calculus of operations and the rise of abstract algebra. *Archive for the History of Exact Sciences* 8.

Laita, L. M. 1977. The influence of Boole's search for a universal method in analysis on the creation of his logic. *Annals of Science* 163–176.

Laita, L. M. 1979. Influences on Boole's logic, the controversy Hamilton-De Morgan. *Annals of Science* 45–65.

Laita, L. M. 1980. Boolean algebras and its extralogical sources. *History and Philosophy of Logic* 37–60.

Langley, P. W.; Zytkow, J. M.; Simon, H. A.; and Bradshaw, G. L. 1987. *Scientific Discovery. Computational Explorations of the Creative Processes.* MIT Press.

Lenat, D. B. 1982. The nature of heuristics. *Artificial Intelligence* 19:189–249.

Oliver, J. E. 1991. *The Incomplete Guide to the Art of Discovery.* New York: Columbia University Press.

Panteki, M. 1991. *Relationships between Algebra, Differential Equations and Logic in England, 188-1860.* Ph.D. Dissertation, Council for National Academic Awards. UK.

Shen, W. M. 1990. Functional transformations in AI discovery systems. *Artificial Intelligence* 41:257–272.