A Surprise-based Qualitative Calculus

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Abstract

This paper introduces a qualitative ranking function that uses signed integers to describe the surprise associated with the occurrence of events. The measure introduced, κ^{++}, is based on the κ calculus but differs from it in that its semantics enable an explicit representation of complements. As a result, the κ^{++} is more capable of enforcing probability theory-like constraints to carry on reasoning.

Introduction

Qualitative formalisms have been proposed to represent uncertain knowledge in lieu of or in conjunction with quantitative methods to avoid the commitment to a full specification of the numerical values associated with the probabilities of beliefs. Such formalisms are used where full precision is undue or unattainable (see (Parsons 2001) for a comprehensive review).

Among the varieties of qualitative frameworks, the kappa (κ) calculus (Spohn 1990; Goldszmidt & Pearl 1996) abstracts probabilities of beliefs by capturing the order of magnitude of the probability of a proposition instead of its numerical value. The resulting measure, $\kappa(.)$, of a belief state of a proposition is a non-negative integer that provides an indication of the incremental degree of surprise associated with the proposition and is used to rank beliefs accordingly. Using this measure one maybe able to, for instance, speak about some event *A* being more likely than another event *B* using the comparison $\kappa(A) < \kappa(B)$. The $\kappa(.)$ measure is used in a calculus designed to reason deductively about defeasible beliefs.

Despite the fact that the $\kappa(.)$ calculus comes bundled with a set of facets that enable Bayesian-like reasoning, the way complements are defined is not intuitive, which makes dealing with them a difficult process. More specifically, one cannot find a property that defines a strong relationship between the $\kappa(.)$ value of a proposition and that of its negation. The only property pertaining to this issue is:

$$\kappa(A) = 0 \text{ or } \kappa(\neg A) = 0$$

Which derives from the analogous property of probability theory: $p(A) + p(\neg A) = 1$ (Goldszmidt & Pearl 1996).

In addition to the fact that the property given admittedly works only for $\kappa(.)$ values that satisfy the property $p(A) = \epsilon^{\kappa}$ such that ϵ is infinitesimally close to zero (Darwiche & Goldszmidt 1994), we find that the property is not sufficient to define the semantics of negation in a calculus to reason about defeasible beliefs. More specifically, if for example $\kappa(A) = 2$ for some belief state A, there is no rule that guides the assignment of a value to $\kappa(\neg A)$ if the $\kappa(.)$ function does not satisfy the property listed above. As a remedy, the $\kappa(.)$ calculus provides for constraints that enforce the otherwise alien rules at the cost of NP-hard reasoning (Goldszmidt & Pearl 1996).

In this paper, we propose $\kappa^{++}(.)$, a qualitative measure of surprise that is semantically capable of handling concepts the $\kappa(.)$ calculus is not capable of.

The κ^{++} Function

$\kappa^{++}(.)$ can be understood as a function which ranks events according to the surprise associated with finding that the event has occurred. It returns a signed integer whose value and sign carry the following semantics:

- *Positive*: $\kappa^{++}(x_r) > 0$ implies that the occurrence of the event x_r indicates a surprise. Moreover, the larger the value of $\kappa^{++}(x_r)$, the more surprising x_r is.
- *Zero*: $\kappa^{++}(x_r) = 0$ represents the most normal world in which the events x_r and $\neg x_r$ are both likely to occur.
- *Negative*: $\kappa^{++}(x_r) < 0$ indicates that the occurrence of x_r is more likely than unlikely. In this case the occurrence of $\neg x_r$ indicates a surprise. Moreover, the larger the magnitude of $\kappa^{++}(x_r)$, the more likely the event.

The above semantics permit a correspondence between κ^{++} and linguistic quantifiers such as the one given in the example below.

.	.
x_r is strongly believed	$\kappa^{++}(x_r) = -2$
x_r is believed	$\kappa^{++}(x_r) = -1$
both x_r and $\neg x_r$ are possible	$\kappa^{++}(x_r) = 0$
$\neg x_r$ is believed	$\kappa^{++}(x_r) = 1$
$\neg x_r$ is believed	$\kappa^{++}(x_r) = 2$
.	.

Relation to Quantitative Surprise Measures

The $\kappa^{++}(.)$ can be interpreted as order of magnitude approximation to the numerical value corresponding to the Weaver index of an event (Weaver 1948). For an event X that has a total I values, the numerical surprise associated with X having specific value x_r is calculated using the Weaver index by dividing the expected value of probability of X by the probability $p(X = x_r)$ as follows:

$$W(x_r) = \frac{\sum_{i=1}^{I} p(x_i)^2}{p(x_r)} \quad (1)$$

As a result, an order of magnitude abstraction of $W(x_r)$, namely $\kappa^{++}(x_r)$, constraints $W(x_r)$ as follows:

$$\epsilon < \frac{W(x_r)}{\epsilon^{\kappa^{++}(x_r)}} \leq 1 \quad (2)$$

Where ϵ is a small positive number.

The interpretation given above can be understood if the probability of an event is represented by a polynomial of ϵ's. In this sense, let $\chi_{x_r}^{n}$ be the polynomial representation of $p(x_r)$, and for every other value x_i of X, let $\chi_{x_i}^{\beta_i}$ denote the polynomial corresponding to $p(x_i)$, with n and β_i being the minimum powers of ϵ in the polynomials respectively. According to equation (1), $W(x_r)$ becomes:

$$W(x_r) = \frac{\sum_{i=1}^{I} p(x_i)^2}{p(x_r)} = \frac{\sum_{i=1}^{I} (\chi_{x_i}^{\beta_i})^2}{\chi_{x_r}^{n}}$$

Since all the polynomials are to the base ϵ, it is possible to add the terms that have equal exponents. This makes the above summation:

$$\frac{\alpha_1 \epsilon^{2\beta_1} + ... + \alpha_I \epsilon^{2\beta_I} + \alpha_{I+1} \epsilon^{2\phi_1} + ... + \alpha_l \epsilon^{2\phi_k}}{\chi_{x_r}^{n}}$$

$\forall \beta_i, 1 \leq i \leq I, \alpha_i \epsilon^{2\beta_i}$ is a term whose power is a candidate to be the minimum power of the polynomial representing $\sum_{i=1}^{I} (\chi_{x_i}^{\beta_i})$ (i.e. the most significant term) as each $2\beta_i$ is the minimum power of $(\chi_{x_i}^{\beta_i})^2$. The ϕ terms in the equation above are non-minimum terms and therefore, their number $(k = l - (I + 1))$ and values are irrelevant for our purpose.

Let m be such term, i.e. $m = \beta_i$ is the minimum of the minimum powers of the polynomials $2\beta_i$. $W(x_r)$ can now be represented only in terms of polynomials as:

$$W(x_r) = \frac{\chi_{x_i}^{2m}}{\chi_{x_r}^{n}}$$

According to equation (2), $\kappa^{++}(x_r)$ can be seen as the order of magnitude abstraction of $W(x_r)$, which implies that $\kappa^{++}(x_r) = 2m - n$, where m is the minimum of all minimum powers in the polynomial $p(x_i)$, $1 \leq i \leq I$, and n is the minimum power in $p(x_r)$.

This interpretation justifies the semantics of $\kappa^{++}(.)$ as large positive values imply a greater difference between $2m$ and n, and as a result, a greater surprise associated with the event. Similarly, the larger the magnitude of a negative $\kappa^{++}(.)$, the larger the difference between $2m$ and n (with $n > 2m$ in this case), and as a result, the more possible the event is compared to other events in the distribution.

The Semantics of Complements

The representation given in the previous section assigns the value $\kappa^{++}(.) = 0$ as the most normal world in which the event and its complement are both possible. In other words, $\kappa^{++}(.) = 0$ is the cutoff value that separates surprising events from non-surprising ones. This enables building constraints on the $\kappa^{++}(.)$ values of complements as given in lemma 1 below.

Lemma 1 $\kappa^{++}(x_r) = \mu \Leftrightarrow \kappa^{++}(\neg x_r) = -\mu$

In other words, a complement of a surprising event is equally unsurprising and vice versa.

For a multi-valued variable, this property generalizes to:

$$[\sum_{i=1}^{I} \kappa^{++}(X = x_i)] = 0$$

Which immediately transforms $\kappa^{++}(.)$ from a mere ranking function to a ranking function that is powerful enough to be used in a formulation of a calculus to propagate surprises associated with events.

It is worth noting that the $\kappa(.)$ function provides for the lack of such property by enforcing constraints to achieve a deductively valid closure on its values. Although sound, these constraints deem reasoning with the $\kappa(.)$ function NP-hard (Goldszmidt & Pearl 1996).

Conclusions and Future Work

We introduced $\kappa^{++}(.)$, a qualitative measure of the degree of surprise associated with an event that can be regarded as the order of magnitude abstraction of the numerical surprise associated with the occurrence of the event. The semantics of κ^{++} reflects the fact that it takes into account the distribution to which the event belongs by ranking an event as *surprising* if it is less likely than the expected outcome of the distribution, and *expected* otherwise. The resulting sign-magnitude ranking function enables the definition of complements with respect to $\kappa^{++}(.)$ and further imposing properties governing the assignments of $\kappa^{++}(.)$ values to events.

A framework that uses probability-like rules to propagate κ^{++}'s constitutes our current research.

References

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