Reasoning with Cardinal Directions: An Efficient Algorithm

Xiaotong Zhang and **Weiming Liu** and **Sanjiang Li**[*] and **Mingsheng Ying**
State Key Laboratory of Intelligent Technology and Systems,
Tsinghua National Laboratory for Information Science and Technology,
Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China

Abstract

Direction relations between extended spatial objects are important commonsense knowledge. Recently, Goyal and Egenhofer proposed a formal model, called Cardinal Direction Calculus (CDC), for representing direction relations between *connected* plane regions. CDC is perhaps the most expressive qualitative calculus for directional information, and has attracted increasing interest from areas such as artificial intelligence, geographical information science, and image retrieval. Given a network of CDC constraints, the consistency problem is deciding if the network is realizable by connected regions in the real plane. This paper provides a cubic algorithm for checking consistency of basic CDC constraint networks. As one byproduct, we also show that any consistent network of CDC constraints has a canonical realization in digital plane. The cubic algorithm can also been adapted to cope with disconnected regions, in which case the current best algorithm is of time complexity $O(n^5)$.

Introduction

Representing and reasoning with spatial information is of particular importance in areas such as artificial intelligence (AI), geographical information systems (GISs), robotics, computer vision, image retrieval, natural language processing, etc. While the numerical quantitative approach prevails in robotics and computer vision, it is widely acknowledged in AI and GIS that the qualitative approach is more attractive.

A predominant part of spatial information is represented by relations between spatial objects. Spatial relations are in general classified into three categories: topological, directional, and metric (e.g. size, distance, shape, etc.). The RCC8 constraint language (Randell, Cui, & Cohn 1992) is the principal topological formalism in AI. When restricted to simple plane regions, RCC8 is equivalent to the 9-Intersection Model (Egenhofer 1991), which is very influential in GIS.

Unlike topological relations, there are several competing formal models for direction relations (Frank 1991; Freksa 1992; Balbiani, Condotta, & Fariñas del Cerro 1999). Most of these models approximate a spatial object by a point (e.g. its centroid) or a box. This is too crude in real-world applications such as describing directional information between two countries, say, Portugal and Spain. Recently, (Goyal & Egenhofer 2001) proposes a relation model, called cardinal direction calculus (CDC), for representing direction relations between connected plane regions. In CDC the reference object is approximated by a box, while leaving the primary object unaltered. This calculus has 218 basic relations and in fact it can distinguish among 2004 different configurations involving two objects (Cicerone & di Felice 2004). Due to its expressiveness, CDC has attracted increasing interest from areas such as AI, GIS, and image retrieval.

One basic criterion for evaluating a formal spatial relation model is the proper balance between its representation expressivity and reasoning complexity. While reasoning complexity of the point-based model of direction relations and the box-based model of direction relations have been investigated in depth (see (Ligozat 1998) and (Balbiani, Condotta, & Fariñas del Cerro 1999)), there are few works discussing complexity of reasoning with CDC.

One fundamental reasoning problem with CDC (and any other qualitative calculus) is the consistency problem. Given a network of CDC constraints

$$\mathcal{N} = \{v_i \delta_{ij} v_j\}_{i,j=1}^n \text{ (each } \delta_{ij} \text{ is a CDC relation)} \quad (1)$$

over n spatial variables $v_1, \cdots, v_n$, the consistency problem is deciding if the network $\mathcal{N}$ is realizable by a set of n *connected* regions in the real plane.

Some restricted versions of the consistency problem have been discussed in the literature. (Cicerone & di Felice 2004) discusses the pairwise consistency problem, which decides when $\{v_1 \delta v_2, v_2 \delta' v_1\}$ is consistent for a pair of basic CDC relations (δ, δ'). (Skiadopoulos & Koubarakis 2004) investigates the weak composition problem (Düntsch, Wang, & McCloskey 2001; Li & Ying 2003), which decides in essence when a basic network involving three variables is consistent. The general case was first discussed in (Navarrete, Morales, & Sciavicco 2007), where the authors proposed an $O(n^4)$ algorithm for consistency checking of basic CDC networks. In this paper, we provide a cubic algorithm for checking consistency of basic CDC constraint networks, which was early observed by Navarrete et al. as impossible. In case the basic network is consistent, our algorithm also generates a realization in cubic time.

[*]Corresponding author. Email:lisanjiang@tsinghua.edu.cn

The remainder of this paper proceeds as follows. Section 2 recalls notations and definitions in CDC. We then introduce the canonical solution of a consistent basic network, and propose an intuitive $O(n^4)$ algorithm for consistency checking of basic networks, and then improve it to $O(n^3)$. Conclusions are given in the last section.

Cardinal Direction Calculus

In this section we recall basic notations and definitions in CDC. In this calculus, we concern ourselves with bounded connected plane regions, where a region is a nonempty regular closed subset. We assume a predefined orthogonal basis in the plane and only consider rectangles or boxes two sides of which are parallel to the axes of the orthogonal basis. For a bounded set b in the real plane, the minimum bounding rectangle (mbr) of b, written $\mathcal{M}(b)$, is defined to be the smallest rectangle which contains b as a part. Write $I_x(b) = [x^-(b), x^+(b)]$ and $I_y(b) = [y^-(b), y^+(b)]$ for, resp., the projection of $\mathcal{M}(b)$ on x- and y-axis. Clearly, $\mathcal{M}(b) = I_x(b) \times I_y(b)$. By extending the four edges of $\mathcal{M}(b)$, we partition the plane into nine tiles, denoted as $NW_b, N_b, NE_b, W_b, O_b, E_b, SW_b, S_b, SE_b$. For ease of representation, we also write in sequence $b_{11}, b_{12}, \cdots, b_{33}$ or $b^1, b^2, \cdots, b^9$ for these tiles. Note that each tile is a region, and the intersection of two tiles is either empty or of dimension lower than two.

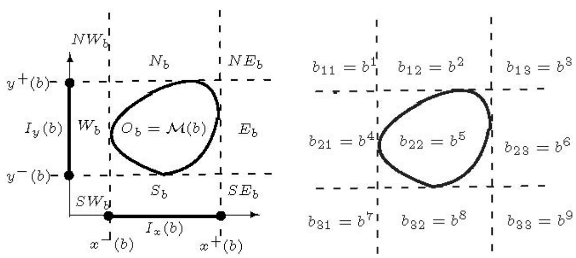


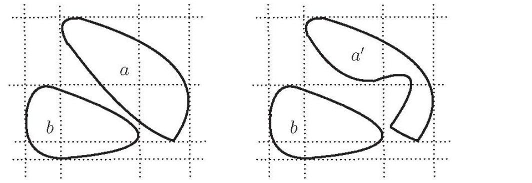
Figure 1: A bounded connected region b and its 9-tiles

Suppose a, b are two bounded connected plane regions. Take b as the reference object, and a as the primary object. The direction of a to b, denoted by $\mathsf{dir}(a, b)$, is encoded in a 3×3 Boolean matrix $[d_{ij}]_{1\leq i,j\leq 3}$, where $d_{ij} = 1$ if and only if $a^\circ \cap b_{ij} \neq \varnothing$, where a° is the interior of a. (see Fig. 2) In this case, we also call $[d_{ij}]$ a valid representation. It is easy to see that not all 3×3 Boolean matrices are valid representations. Goyal and Egenhofer identified all together 218 valid matrices. In other words, there are 218 basic direction relations in CDC, we write $\mathcal{B}_{dir}$ for this set. In what follows we make no distinction between a basic direction relation and its matrix representation.

For a basic relation δ in $\mathcal{B}_{dir}$, we observe that δ may have more than one 'converse.' This means, for constraint $v_1\delta_{12}v_2$, there may exist two different δ_{21} such that $\{v_1\delta_{12}v_2, v_2\delta_{21}v_1\}$ is consistent (see e.g. Fig. 2).

Interval Algebra and CDC relations

If the CDC basic relations δ_{12} and δ_{21} between v_1 and v_2 are known, then the CDC basic relations between their mbrs can



$\mathsf{dir}(b,a) = \mathsf{dir}(b,a') = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $\mathsf{dir}(a,b) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\mathsf{dir}(a',b) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Figure 2: Illustrations of basic CDC relations

be represented by a pair of interval relations (Allen 1983).

Table 1 gives the definitions of basic interval relations.

Table 1: Basic interval relations between two intervals $h = [h^-, h^+], l = [l^-, l^+]$.

Relation	Symb.	Conv.	Meaning
precedes	p	pi	$h^+ < l^-$
meets	m	mi	$h^+ = l^-$
overlaps	o	oi	$h^- < l^- < h^+ < l^+$
starts	s	si	$h^- = l^- < h^+ < l^+$
during	d	di	$l^- < h^- < h^+ < l^+$
finishes	f	fi	$l^- < h^- < h^+ = l^+$
equals	eq	eq	$h^- = l^- < h^+ = l^+$

Write $I_x(v_i)$ $(i = 1, 2)$ for the projection of v_i onto x-axis. Since v_i is interpreted as a connected region, $I_x(v_i)$ is a variable ranging over the set of intervals. Write ι^x_{12} for the possible interval relations between $I_x(v_1)$ and $I_x(v_2)$. Similar interpretation holds for ι^x_{21}. Then we have

Lemma 1. *For two variables v_1, v_2 ranging over connected regions, we have*

- *ι^x_{21} is the converse of ι^x_{12}.*
- *ι^x_{12} is empty or a basic interval relation in $\{\mathsf{o}, \mathsf{s}, \mathsf{d}, \mathsf{f}, \mathsf{eq}, \mathsf{fi}, \mathsf{di}, \mathsf{si}, \mathsf{oi}\}$, or $\iota^x_{12} = \mathsf{p} \cup \mathsf{m}$ or $\iota^x_{12} = \mathsf{pi} \cup \mathsf{mi}$.*

Similar notations and conclusions hold for ι^y_{12} and ι^y_{21}. The CDC relation between $\mathcal{M}(v_1)$ and $\mathcal{M}(v_2)$ is then the rectangle relation $\iota^x_{12} \otimes \iota^y_{12}$, e.g. in Fig. 2, $\mathsf{dir}(\mathcal{M}(b), \mathcal{M}(a)) = \mathsf{o} \otimes \mathsf{o}$.

The next result shows that each consistent basic interval network has a canonical solution in a certain sense.

Theorem 1. *Suppose $\mathcal{N} = \{v_i\lambda_{ij}v_j\}$ is a basic interval network. If $\mathcal{N}$ is consistent, then it has a unique solution $\mathfrak{h} = \{h_i\}_{i=1}^n$ which satisfies the following conditions, where $h_i = [x_i^-, x_i^+]$.*

- *x_i^- and x_i^+ are integers between 0 and $2n - 1$.*
- *There exists some i such that $x_i^- = 0$.*
- *(compactness) For any $0 \leq m \leq n^* = \max\{x_i^+\}_{i=1}^n < 2n$, there is some i such that $x_i^- = m$ or $x_i^+ = m$.*

Proof. Suppose $\{l_i\}$ is a solution of $\mathcal{N}$, where $l_i = [l_i^-, l_i^+]$. Write $\alpha_0 < \alpha_1 < \cdots < \alpha_{n^*}$ for the ordering of $\{l_i^-, l_i^+\}_{i=1}^n$. Define $f, g : \{1, \cdots, n\} \to \{0, 1, \cdots, n^*\}$

as $f(i) = k$ if $l_i^- = \alpha_k$ and $g(i) = k$ if $l_i^+ = \alpha_k$. Let $h_i = [f(i), g(i)]$. Since only the ordering of endpoints of intervals in a solution matters, it is easy to see that $\{h_i\}_{i=1}^n$ is a solution of $\mathcal{N}$ that satisfies all the conditions given above. Such a solution is clearly unique. □

We call $\mathfrak{h}$ the *canonical interval solution* of $\mathcal{N}$.

Sketch of our approach

For a basic constraint network $\mathcal{N} = \{v_i \delta_{ij} v_j\}$, where each $\delta_{ij} \in \mathcal{B}_{dir}$, we sketch a method for checking the consistency of $\mathcal{N}$. The idea is trying to construct a solution that is canonical in a sense similar to that for interval network. To this end, we introduce the concept of digital solution.

Definition 1 (pixel, digital region, digital solution)**.** A *pixel* is a rectangle $p_{ij} = [i, i+1] \times [j, j+1]$, where i, j are integers. A region a is *digital* if a is composed of pixels, i.e. $p_{ij} \cap a^\circ \neq \varnothing$ iff $p_{ij} \subseteq a$. A solution $\mathfrak{a} = \{a_i\}$ of a basic CDC network is *digital* if each a_i is a digital region.

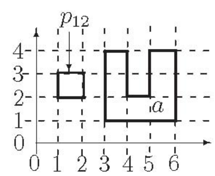


Figure 3: A pixel and a digital region a

Suppose δ_{ij} and δ_{ji} are consistent for any i, j. This implies in particular that the projected interval relations ι_{ij}^x and ι_{ij}^y are nonempty. Recall that ι_{ij}^x and ι_{ij}^y may be disjunctive. We define

$$\rho_{ij}^x = \iota_{ij}^x \setminus (\mathsf{m} \cup \mathsf{mi}) \quad (2)$$

$$\rho_{ij}^y = \iota_{ij}^y \setminus (\mathsf{m} \cup \mathsf{mi}) \quad (3)$$

Then ρ_{ij}^x and ρ_{ij}^y are basic interval relations. See Fig. 4 for an example.

Write $\mathcal{N}_x$ and $\mathcal{N}_y$, resp., for the basic interval networks $\{I_x(v_i)\rho_{ij}^x I_x(v_j)\}$ and $\{I_y(v_i)\rho_{ij}^y I_y(v_j)\}$. We shall prove (Lemma 5) that $\mathcal{N}$ has a solution only if both $\mathcal{N}_x$ and $\mathcal{N}_y$ have solutions. So if either $\mathcal{N}_x$ or $\mathcal{N}_y$ is inconsistent, that $\mathcal{N}$ must be inconsistent.

In case that both $\mathcal{N}_x$ and $\mathcal{N}_y$ are consistent, suppose $\mathcal{I} = \{I_i\}$ and $\mathcal{J} = \{J_i\}$ are their canonical interval solutions given by Thm. 1, where $I_i = [x_i^-, x_i^+]$, $J_i = [y_i^-, y_i^+]$.

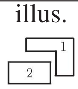
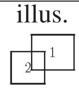
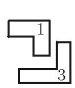
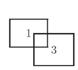
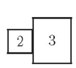
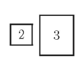
(i,j)	δ_{ij}	δ_{ji}	illus.	$\rho_{ij}^x \otimes \rho_{ij}^y$	illus.
(1,2)	$\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\1&1&0\\1&1&0\end{bmatrix}$		oi ⊗ oi	
(1,3)	$\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&1\\0&1&1\end{bmatrix}$		o ⊗ oi	
(2,3)	$\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}$		p ⊗ d	

Figure 4: A basic CDC network

Write $n_x = \max\{x_i^+\} < 2n$, $n_y = \max\{y_i^+\} < 2n$. We further show that (Lemma 5), if $\mathcal{N}$ is consistent, then we can find a digital solution $\mathfrak{a} = \{a_i\}$ of $\mathcal{N}$ in $T = [0, n_x] \times [0, n_y]$ such that $r_i \equiv I_i \times J_i$ is the mbr of a_i. Therefore, we need only search in a restricted space that contains at most $4n^2$ pixels.

For the network specified in Fig. 4, we have

$$x_2^- < x_1^- < x_2^+ < x_3^- < x_1^+ < x_3^+ \quad (4)$$

$$y_3^- < y_2^- < y_1^- < y_2^+ < y_3^+ < y_1^+ \quad (5)$$

The canonical interval solutions of $\mathcal{N}_x$ and $\mathcal{N}_y$ are illustrated in Fig. 5.

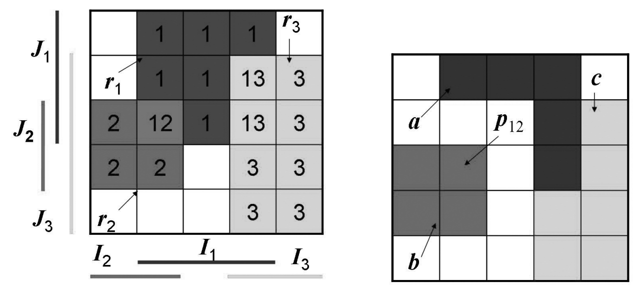


Figure 5: Canonical interval solutions and maximal solution

By constraints in $\mathcal{N}$, some pixels cannot appear in a_i. Suppose $\{a, b, c\}$ is a solution of the network described in Fig. 4. Note that the (2,2)-entry of δ_{12} is 0, which is possible only if $a \cap \mathcal{M}(b) = \varnothing$. This excludes pixel p_{12} from a (see Fig. 5).

By deleting all pixels in r_i that are not allowed, we obtain a subset b_i of r_i, which may be disconnected in general, but happens to be connected for our example (see Fig. 5). We then compute the maximally connected components (*mcc*s) of b_i in S. If $\mathcal{N}$ is consistent, Lemma 5 will prove that $\mathcal{N}$ has a digital solution $\{o_i\}$ in $[0, n_x] \times [0, n_y]$ s.t. $m(o_i) = r_i$. If r_i is not the mbr of any mcc of b_i, $\mathcal{N}$ must be inconsistent.

In the other case, if each b_i has an mcc c_i such that $\mathcal{M}(c_i) = r_i$, we choose $\{c_i\}$ as a possible solution of $\mathcal{N}$ and check if it satisfies all constraints in $\mathcal{N}$.[1] If the answer is negative, then we claim that $\mathcal{N}$ is inconsistent, and consistent otherwise. As for our running example, $\{a, b, c\}$ in Fig. 5 (right) provides a solution that is maximal in a certain sense (see § 3).

Canonical Solution

Suppose $\mathcal{N} = \{v_i \delta_{ij} v_j\}_{i,j=1}^n$ is a consistent network of basic CDC constraints. We prove that $\mathcal{N}$ has a digital solution which is *canonical* in certain sense.

To this end, we first show that $\mathcal{N}$ has a meet-free solution in the following sense.

Definition 2 (meet-free solution)**.** A solution $\mathfrak{a} = \{a_i\}_{1 \leq i \leq n}$ of $\mathcal{N}$ is *meet-free* if for any i, j, $I_x(a_i)$ does not meet $I_x(a_j)$, and $I_y(a_i)$ does not meet $I_y(a_j)$.

[1] We note that such an mcc is unique since two such digital regions must share one pixel.

Suppose $\mathfrak{a} = \{a_i\}_{1 \le i \le n}$ is a solution of $\mathcal{N}$. We show $\mathcal{N}$ has a meet-free solution. To this end, we introduce the idea of *regularization*.

Write $m_i = [x_i^-, x_i^+] \times [y_i^-, y_i^+]$ for the mbr of a_i. Denote $e^- = \inf\{x_i^- : 1 \le i \le n\}$, $e^+ = \sup\{x_i^+ : 1 \le i \le n\}$, $f^- = \inf\{y_i^- : 1 \le i \le n\}$, $f^+ = \sup\{y_i^+ : 1 \le i \le n\}$. Let $S = [e^-, e^+] \times [f^-, f^+]$. Extending edges of each m_i until the boundary of S is met, we partition S into small cells. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_{n_x}$ is the ordering of real numbers in $\{x_i^-, x_i^+ : 1 \le i \le n\}$, and $\beta_0 < \beta_1 < \cdots < \beta_{n_y}$ is the ordering of real numbers in $\{y_i^-, y_i^+ : 1 \le i \le n\}$. Denote c_{ij} for the cell $[\alpha_i, \alpha_{i+1}] \times [\beta_j, \beta_{j+1}]$, and write C for the set of these cells. For each i, let $a_i^r = \bigcup\{c \in C : c \cap a_i^\circ \neq \varnothing\}$. Fig. 6 gives an example.

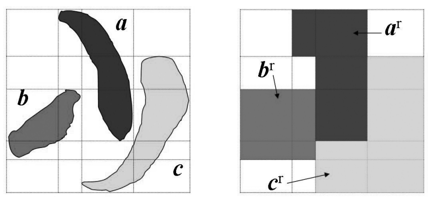


Figure 6: Illustration of regularization

Lemma 2. *$\mathfrak{a}^r = \{a_i^r\}$ is also a solution of $\mathcal{N}$.*

Proof. It is clear that each a_i^r is connected, and $\mathcal{M}(a_i^r) = \mathcal{M}(a_i)$. Then it is straightforward to show that $\mathsf{dir}(a_i, a_j) = \mathsf{dir}(a_i^r, a_j^r)$ for any i, j. $\square$

We call $\mathfrak{a}^r$ the *regularization* of $\mathfrak{a}$.

We next show how to obtain a meet-free solution step by step. Suppose $m_i = [x_i^-, x_i^+] \times [y_i^-, y_i^+]$ and $m_j = [x_j^-, x_j^+] \times [y_j^-, y_j^+]$ meet at x direction, i.e. $x_i^+ = x_j^-$. Suppose $x_i^+ = \alpha_k$ and call α_k an x-meet point. Then $x_i^+ = x_j^- = \alpha_k < \alpha_{k+1} \le x_j^+$. Write $\alpha^* = (\alpha_k + \alpha_{k+1})/2$. The line $x = \alpha^*$ divides each cell c_{kl} into two equal parts, written in order c_{kl}^- and c_{kl}^+. For each $1 \le s \le n$ and each $0 \le l < n_y$, if $c_{kl} \subseteq a_s$ but $c_{k-1,l} \not\subseteq a_s$ then delete c_{kl}^- from a_s. The remaining part of each a_s, written as b_s, is still connected, and it is straightforward to show that $\mathfrak{b} = \{b_s\}$ is also a solution of $\mathcal{N}$. Such a modification introduces no new meet points. Continuing this process for at most n times, we will have a solution that has no x-meet points. The same modification can be applied to y-meet points. In this way we obtain a meet-free solution. An example is shown in Fig. 7.

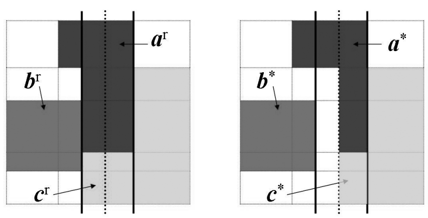


Figure 7: Illustration of meet-freeing

In conclusion, we have the following lemma:

Lemma 3. *Each consistent CDC network has a meet-free solution.*

So we can safely assume that $\mathfrak{a}$ is a meet-free solution that is regularized, i.e. $\mathfrak{a}^r = \mathfrak{a}$. We next construct a digital solution.

As in the paragraph immediately above Lemma 2, we write $m_i = [x_i^-, x_i^+] \times [y_i^-, y_i^+]$ for the mbr of a_i. Let S, α_i, β_j, n_x, n_y as before, and let $c_{ij} = [\alpha_i, \alpha_{i+1}] \times [\beta_j, \beta_{j+1}]$, and write C for the set of these cells. Since $\mathfrak{a}$ is regularized, each a_i is the union of all cells in C that are contained in a_i.

We next show that $\mathcal{N}$ has a digital solution in $T = [0, n_x] \times [0, n_y]$. For $1 \le s \le n$, define a subset b_s of T as follows: a pixel p_{ij} is in b_s iff the cell c_{ij} in C is contained in a_s. Clearly, b_s is a connected region. By definition we have $\mathsf{dir}(a_i, a_j) = \mathsf{dir}(b_i, b_j)$ for any i, j. Therefore the assignment $\mathfrak{b} = \{b_s\}$ is also a solution of $\mathcal{N}$.

The digitalization of the solution $\{a^*, b^*, c^*\}$ in Fig. 7 (right) is indeed the one given in Fig. 5 (right).

We observe that $\mathfrak{b}$ is canonical in the following sense.

Definition 3 (canonical solution)**.** A digital solution $\mathfrak{a} = \{a_i\}_{1 \le i \le n}$ of $\mathcal{N}$ is said *canonical* if:

- $\mathfrak{a}$ is meet-free.
- for all i , x_i^- , x_i^+ , y_i^- and y_i^+ are all integers, and $\inf\{x_i^-\}_{i=1}^n = \inf\{y_i^-\}_{i=1}^n = 0$.
- $\mathfrak{a}$ is compact, i.e.
  - for each integer $0 \le m \le \sup\{x_i^+\}_{i=1}^n$, m is an end-point of some interval $I_x(a_i)$;
  - for each integer $0 \le m \le \sup\{y_i^+\}_{i=1}^n$, m is an end-point of some interval $I_y(a_i)$;

where $I_x(a_i) = [x_i^-, x_i^+]$ and $I_y(a_i) = [y_i^-, y_i^+]$.

Lemma 4. *$\mathfrak{b} = \{b_i\}$ is a canonical solution.*

Moreover, since $\mathfrak{b}$ is meet-free, $I_x(b_i)$ cannot meet $I_x(b_j)$ for any two i, j. This implies that $\{I_x(b_i)\}$ is a solution to the basic interval network $\mathcal{N}_x = \{I_x(v_i) \rho_{ij}^x I_x(v_j)\}$, where ρ_{ij}^x is defined by Eq. 2. Similar conclusion holds for $\mathcal{N}_y$.

As a consequence, we have

Lemma 5. *If $\mathcal{N}$ is consistent, then it has a canonical solution $\mathfrak{a} = \{a_i\}$ such that $\{I_x(a_i)\}$ and $\{I_y(a_i)\}$ are, resp., the canonical interval solutions of $\mathcal{N}_x$ and $\mathcal{N}_y$.*

By the uniqueness of canonical interval solution (Theorem 1), we have the following

Lemma 6. *Suppose $\mathfrak{b} = \{b_i\}$ and $\mathfrak{b}' = \{b_i'\}$ are two canonical solutions of $\mathcal{N}$. Then $I_x(b_i) = I_x(b_i')$, $I_y(b_i) = I_y(b_i')$, and $\mathcal{M}(b_i) = \mathcal{M}(b_i')$.*

This shows that two canonical solutions have the same mbrs. Is there a maximal canonical solution? The following lemma asserts this is true.

Lemma 7. *Suppose $\mathfrak{b} = \{b_i\}$ and $\mathfrak{b}' = \{b_i'\}$ are two canonical solutions of $\mathcal{N}$. Then $\mathfrak{c} = \{b_i \cup b_i'\}$ is also a canonical solution of $\mathcal{N}$.*

Proof sketch. Since $\mathcal{M}(b_i) = \mathcal{M}(b_i')$, we know b_i shares a pixel with b_i', therefore $c_i \equiv b_i \cup b_i'$ is connected. It is then straightforward to show that $\mathsf{dir}(c_i, c_j) = \mathsf{dir}(b_i, b_j)$. $\square$

The maximal canonical solution is then the union of all canonical solutions, which is apparently unique.

Corollary 1. *If a basic CDC network $\mathcal{N}$ is consistent, then it has a unique maximal canonical solution.*

A Consistency Checking Algorithm

In this section, we describe our algorithm for checking consistency of basic CDC networks. As in the last section, we suppose $\mathcal{N} = \{v_i \delta_{ij} v_j\}$ is a basic CDC network.

For ease of representation, in this section we write a CDC basic relation $\delta = (d_{st})_{1\leq s,t\leq 3}$ as a 9-tuple $\delta = (d^\phi)_{1\leq\phi\leq 9}$, where $\phi = 3(s-1)+t$. We also write b^ϕ for the tile b_{st}. (see Fig. 1 (right))

Step 1. Projected basic interval networks

For any CDC basic constraints $\{x_1\delta_{12}x_2, x_2\delta_{21}x_1\}$, the x- and y-projected interval relations ρ^x_{12} and ρ^y_{12} can be computed in constant time. So the projected interval networks $\mathcal{N}_x$ and $\mathcal{N}_y$ can be constructed in $O(n^2)$ time. If $\mathcal{N}_x$ or $\mathcal{N}_y$ is inconsistent, which can be checked in cubic time, then $\mathcal{N}$ is inconsistent.

Step 2. Canonical interval solutions

Suppose $\mathcal{N}_x$ and $\mathcal{N}_y$ are consistent. Their canonical solutions $\{[x_i^-, x_i^+]\}$ and $\{[y_i^-, y_i^+]\}$ can be constructed in cubic time. Recall $x_i^-, x_i^+, y_i^-, y_i^+$ are integers between 0 and $2n-1$. Write $n_x = \max\{x_i^+\}$ and $n_y = \max\{y_i^+\}$. Define $m_i = [x_i^-, x_i^+] \times [y_i^-, y_i^+]$. By Corollary 1, if $\mathcal{N}$ is consistent, it has a unique maximal canonical solution $\mathfrak{a} = \{a_i\}$, where $a_i \subseteq [0, n_x] \times [0, n_y]$ and $\mathcal{M}(a_i) = m_i$ for each i.

Step 3. Excluding impossible pixels

Suppose $\mathcal{N}$ is consistent and $\mathfrak{a} = \{a_i\}$ is its maximal canonical solution. Note that if $d^\phi_{ij} = 0$, then $b_i^\circ \cap m_j^\phi = \varnothing$, where $1 \leq i,j \leq n$ and $1 \leq \phi \leq 9$. Therefore, all pixels contained in m_j^ϕ cannot appear in a_i, i.e., these pixels should be excluded from m_i.

Define

$$b_i^* = \overline{m_i \setminus \bigcup\{m_j^\phi : d^\phi_{ij} = 0\}}, \tag{6}$$

where $\overline{o}$ is the closure of o. To compute b_i^*, an intuitive method is checking for each pixel p in m_i, and each tile m_j^ϕ of any reference object m_j, whether p is in m_j^ϕ. This requires cubic time for each b_i^*, and hence $O(n^4)$ time in total. Later, we will show this can be improved to $O(n^3)$.

Step 4. Maximally connected components

We further compute the maximally connected components (mccs) of each b_i^*. Note that if it is consistent, $\mathcal{N}$ has a unique maximal canonical solution $\{a_i\}$ such that $\mathcal{M}(a_i) = m_i$. Consequently, if m_i is not the mbr of any mcc of b_i^*, then $\mathcal{N}$ must be inconsistent. Otherwise, there must be a unique mcc, written c_i, of b_i^* such that $\mathcal{M}(c_i) = m_i$.

Applying a general Breadth-First Search algorithm, we can find all mccs of b_i^* in $O(n^2)$ time. Determining if the mbr of an mcc c is m_i requires at most $O(n^2)$ time.

Step 5. Checking a possible solution

The last step is then checking if $\{c_i\}$ is a solution of $\mathcal{N}$. Note that if the answer is yes, then $\{c_i\}$ is the maximal canonical solution of $\mathcal{N}$.

For each pair c_i and c_j, we should check if $\mathsf{dir}(c_i, c_j) = \delta_{ij}$. In other words, we should check for each $1 \leq \phi \leq 9$ with $d^\phi_{ij} = 1$, whether $c_i^\circ \cap m_j^\phi \neq \varnothing$. This is because, if $d^\phi_{ij} = 0$, then $c_i^\circ \cap m_j^\phi \subseteq b_i^\circ \cap m_j^\phi = \varnothing$, the condition is already satisfied.

Recall c_i is an mcc of b_i and $\mathcal{M}(c_i) = m_i$. If $c_i \subseteq m_j^\phi$ then clearly $c_i^\circ \cap m_j^\phi \neq \varnothing$. Suppose $c_i \not\subseteq m_j^\phi$. Then $c_i^\circ \cap m_j^\phi \neq \varnothing$ iff c_i contains a boundary pixel of m_j^ϕ. Write $m_j^\phi = [x^-, x^+] \times [y^-, y^+]$. We need only check for each pixel p_{kl} with $(k,l) \in H_1 \cup H_2$ whether $p_{kl} \subseteq c_i$, where

$$\begin{aligned} H_1 &= \{(k,l) : k \in \{x^-, x^+ - 1\} \text{ and } y^- \leq l < y^+\}, \\ H_2 &= \{(k,l) : x^- \leq k < x^+ \text{ and } l \in \{y^-, y^+ - 1\}\}. \end{aligned}$$

Since $H_1 \cup H_2$ contains $O(n)$ pixels and checking if a pixel is contained in c_i needs constant time, $\mathsf{dir}(c_i, c_j) = \delta_{ij}$ can be checked in $O(n)$ time.

In conclusion, we can determine in cubic time whether $\{c_i\}$ is a solution of $\mathcal{N}$. Now, since only Step 3 needs at most $O(n^4)$ time, the algorithm determines the consistency of a basic CDC network in $O(n^4)$ time.

Improvement and Discussion

In this section, we first improve the algorithm to cubic, then discuss related work.

A cubic algorithm for excluding impossible pixels

We now present a cubic algorithm for computing b_i^* in Eq. 6. We first observe that m_i and $m_j^\phi \cap T$ are all digital regions in $T = [0, n_x] \times [0, n_y]$, which can be represented as[2] $(n_x + 1) \times n_y$ Boolean matrices in a natural way. Suppose $m = [x^-, x^+] \times [y^-, y^+]$ is a rectangle. We define the Boolean matrix B of m as follows

$$B[k,l] = \begin{cases} 1, & \text{if } x^- \leq k < x^+ \text{ and } y^- \leq l < y^+; \\ 0, & \text{otherwise.} \end{cases}$$

We write B_i and B_j^ϕ for, resp., the Boolean matrix of m_i and m_j^ϕ. The early approach for computing b_i^* can be rephrased as follows. Adding up all B_j^ϕ with $d^\phi_{ij} = 0$ we obtain an integer matrix P. This requires cubic time. Define Q as

$$Q[k,l] = \begin{cases} 1, & \text{if } P[k,l] = 0 \text{ and } B_i[k,l] = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then Q represents the region b_i^*.

For any matrix N, we define its cumulative matrix as $\mathsf{acc}(N)[k,l] = \Sigma_{t=0}^{k} N[t,l]$ for each k,l. It is clear that $\mathsf{acc}(N)$ can be computed in $O(n^2)$ time, and $\mathsf{acc}(N_1 + N_2) = \mathsf{acc}(N_1) + \mathsf{acc}(N_2)$.

[2] We add an additional column for ease of counting.

We next introduce another matrix representation of m_j^ϕ as

$$A_j^\phi[k,l] = \begin{cases} 1, & \text{if } k = x^- \text{ and } y^- \le l < y^+; \\ -1, & \text{if } k = x^+ \text{ and } y^- \le l < y^+; \\ 0, & \text{otherwise,} \end{cases}$$

where $m_j^\phi = [x^-, x^+] \times [y^-, y^+]$. It is easy to see that $\mathsf{acc}(A_j^\phi) = B_j^\phi$. Adding up all A_j^ϕ we obtain a matrix A. Clearly, $P = \Sigma\{B_j^\phi : d_{ij}^\phi = 0\} = \Sigma\{\mathsf{acc}(A_j^\phi) : d_{ij}^\phi = 0\} = \mathsf{acc}(\Sigma\{A_j^\phi : d_{ij}^\phi = 0\}) = \mathsf{acc}(A)$. So this provides another way for computing the Boolean Matrix representation Q of b_i^*.

The trick is that A, hence b_i^*, can be computed in $O(n^2)$ instead of $O(n^3)$ time.

Algorithm 1 ADDING-UP

Input: All m_j^ϕ with $d_{ij}^\phi = 0$
Output: A matrix A
  Initialize A with a zero matrix
  for each $m_j^\phi = [x^-, x^+] \times [y^-, y^+]$ **do**
    for $l = y^-$ **to** $y^+ - 1$ **do**
      $A[x^-, l] \leftarrow A[x^-, l] + 1$
      $A[x^+, l] \leftarrow A[x^+, l] - 1$
  Output A

Further Discussion

(Navarrete, Morales, & Sciavicco 2007) proposed an $O(n^4)$ algorithm for checking consistency of basic CDC networks, which is based on the following unjustified Helly's Topological Theorem (HTT):

Let F be a finite family of closed sets in $\mathbb{R}^2$ such that the intersection of every k members of F is a cell, for $k \le 2$, and it is nonempty for $k = 3$. Then $\bigcap F$ is a cell, i.e. homeomorphic to a closed disk.

The example giving in Fig. 8 shows HTT is incorrect.

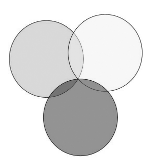


Figure 8: A counter-example of HTT

Our cubic algorithm can also be extended to cope with possibly disconnected regions. In this case, Steps 1-3 remain the same, and in Step 4 we take c_i to be b_i^* and check if $\mathcal{M}(c_i) = m_i$. As for Step 5, we define $M_i[k,l] = \sum_{1 \le p \le k, 1 \le q \le l} c_i[p,q]$, which can be computed in $O(n^2)$ time for each c_i. For any two i, j, we need check for each $1 \le \phi \le 9$ with $d_{ij}^\phi = 1$ whether $c_i^\circ \cap m_j^\phi \ne \varnothing$. Taking advantage of M_i, this costs only constant time. Therefore, Step 5, and thereby the algorithm, can be finished in $O(n^3)$ time.

Conclusion

This paper provided a cubic algorithm for checking consistency of basic CDC networks, which was earlier observed as impossible for connected regions. Extended to possibly disconnected regions, our algorithm also provides a significant improvement to the $O(n^5)$ algorithm of (Skiadopoulos & Koubarakis 2005). Our future work will consider combination of CDC constraints with topological RCC8 constraints (cf. (Li 2007)).

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