A Qualitative Method to Construct Phase Portraits*

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Abstract

We have developed and implemented in the QPORTRAIT program a qualitative simulation based method to construct phase portraits for a significant class of systems of two coupled first order autonomous differential equations, even in the presence of incomplete, qualitative knowledge.

Differential equation models are important for reasoning about physical systems. The field of nonlinear dynamics has introduced the powerful phase portrait representation for the global analysis of nonlinear differential equations.

QPORTRAIT uses qualitative simulation to generate the set of all possible qualitative behaviors of a system. Constraints on two-dimensional phase portraits from nonlinear dynamics make it possible to identify and classify trajectories and their asymptotic limits, and constrain possible combinations. By exhaustively forming all combinations of features, and filtering out inconsistent combinations, QPORTRAIT is guaranteed to generate all possible qualitative phase portraits. We have applied QPORTRAIT to obtain tractable results for a number of nontrivial dynamical systems.

Guaranteed coverage of all possible behaviors of incompletely known systems complements the more detailed, but approximation-based results of recently-developed methods for intelligently-guided numeric simulation [Nishida et al; Sacks; Yip; Zhao]. Combining the strengths of both approaches would better facilitate automated understanding of dynamical systems.

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Intoduction

This report[1] describes a qualitative simulation based method, implemented in the QPORTRAIT program, to construct phase portraits for a significant class of systems of two first order autonomous differential equations. It is a step towards a useful tool for automated reasoning about dynamical systems (i.e. differential equations), and shows that a dynamical systems perspective can give a tractable overview of a qualitative simulation problem.

Differential equations are important for reasoning about physical systems. While nonlinear systems often require complex idiosyncractic treatments, phase potraits have evolved as a powerful tool for global analysis of them. A state of a system is represented by a point in the phase space; change of the system state over time is represented by a trajectory; and a phase portrait is the collection of all possible trajectories of the system.

Phase portraits are typically constructed for exactly specified system instances by intelligently choosing samples of trajectories for numeric simulation and interpreting the results. This has led to recent development of numeric methods based reasoning in the phase space [Nishida et al; Sacks; Yip; Zhao]. These approaches are able to give good approxiamte results.

Based on qualitative simulation [Kuipers 86], and using knowledge of dynamical systems, QPORTRAIT is able to predict all possible phase portraits of incompletely known systems (in the form of qualitative differential equations, QDEs). Starting with a total envisionment [Forbus 84] of a system, QPORTRAIT progressively identifies, classifies, and combines features of the phase portrait, abstracting away uninteresting distinctions, and filtering out inconsistent combinations of features. Exhaustive search and elimination of only provable inconsistencies enable guaranteed coverage of behaviors. This, and the ability to handle incomplete information about systems complement numeric methods based approaches.

QPORTRAIT is currently applicable to systems of two

[1]This report summarizes the work of [Lee 93].

first order autonomous differential equations with nondegenerate fixed points. Various recently developed techniques have been incorporated to deal with qualitative simulation's potential for intractability. As a result, QPORTRAIT is able to produce tractable results for systems with fixed points at landmark values for the phase variables. We have applied QPORTRAIT to obtain tractable results for QDE versions of several well-known nonlinear systems, including a Lienard equation, a van der Pol equation, an undamped pendulum, and a predator-prey system

In the rest of this report, we will first describe the underlying concepts of our work. Then we will describe the steps of our method, followed by an illustration of the steps using a Lienard equation example. Next we will present an argument that our method provides guarantee of coverage, discuss dependencies and limitations, and describe related work. We then end this report with our conclusion.

Underlying Concepts

Phase Portraits

In the phase portrait representation, a state of a system is represented by a point in the system's *phase space*, defined by a set of phase variables of the system. (A set of *phase variables* of a system is a minimal set of variables that fully describes the state of the system.) Change of the system state over time is represented by a *trajectory* in the phase space. A *phase portrait* is the collection of all possible trajectories of the system. The key characteristics of a phase portrait are the *asymptotic limits* of trajectories (i.e. where trajectories may emerge or terminate), and certain bounding trajectories that divide the phase space into stability regions.

For autonomous two-dimensional systems,

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y), \end{cases}$$

asymptotic limits of trajectories can only be one of: fixed points (where the system is stationary), closed orbits (where the system oscillates steadily forever), unions of fixed points and the trajectories connecting them, and points at infinity. Fixed points are either *sinks* (where trajectories only terminate), *sources* (where trajectories only emerge), *saddles* (where trajectories may either emerge or terminate), or *centers* (where trajectories neither emerge or terminate). Bounding trajectories other than closed orbits are associated with saddles, and are called *separatrices*.

In restricting our attention to system with nondegenerate fixed points (which are noncontiguous), local characteristics of fixed points are essentially linear. This means, in particular, that unions of fixed points and the trajectories connecting them can only be unions of saddles and separatrices connecting them. Furthermore, the essentially linear characteristics of saddle points means that exactly one separatrix enters a saddle in either of two opposite directions, and exactly one separatrix exits a saddle in either of two opposite directions.

Reasoning in Qualitative Phase Space[2]

To reason about phase portraits in qualitative phase space, we integrate the *total envisionment* and *behavior generation* approaches in qualitative simulation. A total envisionment [Forbus 84], using a coarse state space representation[3], produces a transition graph of the n-dimensional state space for the QDE of the system in question. This includes all possible qualitative states a system can take on, and possible transitions between them, capturing all possible trajectories, and their asymptotic limits. Behavior generation [Kuipers 86] refines trajectory paths for two purposes: to check for each trajectory that not all behavior refinements of it are provably inconsistent, and to depict detailed trends of cyclic paths. These ideas are further discussed in the next section.

A QDE description of a system may apply to instances of a system that give rise to phase portraits with different local characteristics. For example, a nonlinear oscillator may be overdamped, giving rise to non-spiraling (nodal) trajectories into a sink; partially underdamped, with trajectories spiraling an arbitrary finite number of times around a sink; or totally underdamped, spiraling infinitely many times as it converges to the sink. These trajectories are mutually intersecting, and belong to different phase portraits, but the distinctions are local to the cyclic paths around a particular sink.

In order to arrive at a tractable global view of the set of qualitative phase portraits, we abstract such a local configuration into a *spiral-nodal bundle* of trajectories around a given sink or source [Lee93], representing the bundle with one of the constituents. Other examples of abstracting away detailed distinctions are discussed subsequently.

Steps of QPORTRAIT

The major steps of QPORTRAIT are:

1. envision, through total envisionment, to capture all possible trajectories and their asymptotic limits,
2. identify the asymptotic limits (possible origins and destinations) from the envisionment graph,
3. gather trajectories by exhaustively tracing paths between possible origins and destinations,

[2] Notable earlier work in this area has been done by [Chiu 88], [Lee & Kuipers 88] and [Struss 88].

[3] The value a variable can take on is from a predetermined set of *landmark values* for the variable, or the set of intervals between these landmarks.

4. compose mutually non-intersecting trajectories into phase portraits.

With a few exceptions identified explicitly below, all steps in this analysis have been automated. These techniques are described in more detail in [Lee93].

Capturing all Trajectories

A QDE is first constructed for the system in question. While this process is manually performed, there are often straightforward transformations between functional relationships and QDE constraints. Next, total envisionment captures all possible trajectories and their asymptotic limits. Fixed points are then identified and checked for nondegeneracy[4]. This involves symbolic algebraic manipulation, and is performed manually (though a simple version can be relatively easily implemented). Potentially degenerate fixed points suggest possible bifurcation, and the system needs to be decomposed along these points.

Before proceeding to identify asymptotic limits, the envisionment graph is projected onto the phase plane, and states not giving rise to distinctions in the phase plane are removed. These techniques are described in [Fouché 92] and [Lee93].

Identifying Asymptotic Limits

The complete set of possible asymptotic limits (origins and destinations) of trajectories for autonomous two-dimensional systems with nondegenerate fixed points can be identified from the total envisionment graph.

1. Fixed points are quiescent states in the envisionment graph. Sinks have only predecessors; sources have only successors; saddles have both; and centers have neither.
2. Closed orbits are closed paths in the graph. (Closed paths may also represent inward or outward spirals. These possibilities are distinguished in the next step, gathering trajectories.)
3. Separatrices are paths connecting to saddle points. The union of saddle points and separatrices connecting them (homoclinic and heteroclinic orbits) can also be asymptotic limits of trajectories.
4. Points at infinity that are asymptotic limits have either ∞ or $-\infty$ as their *qmag*, and either have no predecessors, or have no successors.

Gathering Trajectories

Trajectories are gathered by exhaustively tracing possible paths between origins and destinations, abstracting away unimportant distinctions. Loops representing chatter [Kuipers & Chiu 87], and topologically equivalent paths (i.e. sets of mutually homotopic trajectories), are abstracted away and replaced by their simplest representative path.

[4] This is done by checking to see that the eigenvalues of the Jacobian matrix of the system at the fixed points have nonzero real parts.

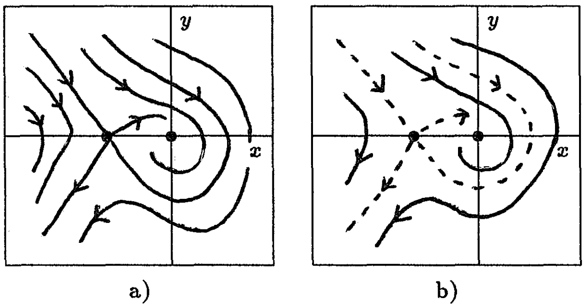


Figure 1: Phase portraits of the Lienard equation: a) from [Brauer & Nohel 69] and [Sacks 90], and b) from QPORTRAIT.

When one of the resulting trajectories contains a cyclic path in the envisionment graph, its qualitative description is refined through behavior generation in order to determine possible trends of the cycle (spiral inward, spiral outward, and/or periodic). Envisionment-guided simulation [Clancy & Kuipers 92], the energy filter [Fouché & Kuipers 92], and cycle trend extraction [Lee93], are used for this task. Once cycle trends have been established, incomplete cyclic trajectory fragments can be combined in all consistent ways with connecting fragments to form complete trajectories.

Next, trajectories around sinks and sources are analyzed, and spiral-nodal bundles are identified and abstracted. Each trajectory is checked to see that not all behavior refinements are provably inconsistent.

Composing Portraits

Trajectories gathered are first classified as either separatrices, which connect to saddle points (and are bounding trajectories that divide the phase space into stability regions), and flows, which do not. At each saddle, QPORTRAIT composes all possible separatrix sets, each consisting of non-intersecting separatrices with exactly one entering the saddle in each of two opposite directions, and exactly one exiting in each of two opposite directions. The method for enforcing non-intersection of qualitative trajectories is described in [Lee & Kuipers 88].

All possible non-intersecting combinations of separatrix sets between saddle points are then formed, and all possible non-intersecting flows are composed into each combination to form all possible qualitative phase portraits.

A Lienard Equation Example

A particular instance of the Lienard equation takes the form ([Brauer & Nohel 69] pp. 217):

$$x'' + x' + x^2 + x = 0,$$

or equivalently:

$$\begin{cases} x' = y \\ y' = -(x^2 + x) - y. \end{cases}$$

It has an interesting phase portrait, discussed in detail in [Brauer & Nohel 69] pp. 217–220, and used in [Sacks 90] as a main example. Its phase portrait (from [Brauer & Nohel 69] pp. 220) is as shown in Figure 1a. The portarit produced in [Sacks 90] has the same essential qualitative features.

A QDE generalization of this equation has the x^2+x term replaced by a U^+ function[5]:

$$\begin{cases} x' = y \\ y' = -f(x) - y, \quad f \in U^+_{(a,b),(c,0),(0,0)}; a, b < 0; c < a. \end{cases}$$

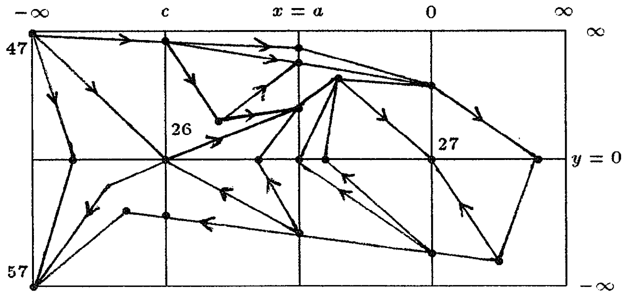
QPORTRAIT is able to produce for this QDE the phase portrait in Figure 1b. This portrait has the same essential features as the one in Figure 1a, though ours is applicable to the QDE. We describe briefly below results of intermediate steps for arriving at this portrait.

Applying total envisionment, projecting the envisionment graph onto the phase plane, and removing states not giving rise to interesting distinctions give the envisionment graph in Figure 2a. The potential asymptotic limits are the fixed points at S-26 which is a saddle, the fixed point at S-27 which is a sink, the closed paths around S-27, the paths connecting S-26 to itself (which are separatrices connecting a saddle to itself), and the points at infinity, S-47 and S-57. They are automatically identified from the graph. Both fixed points are nondegenerate.

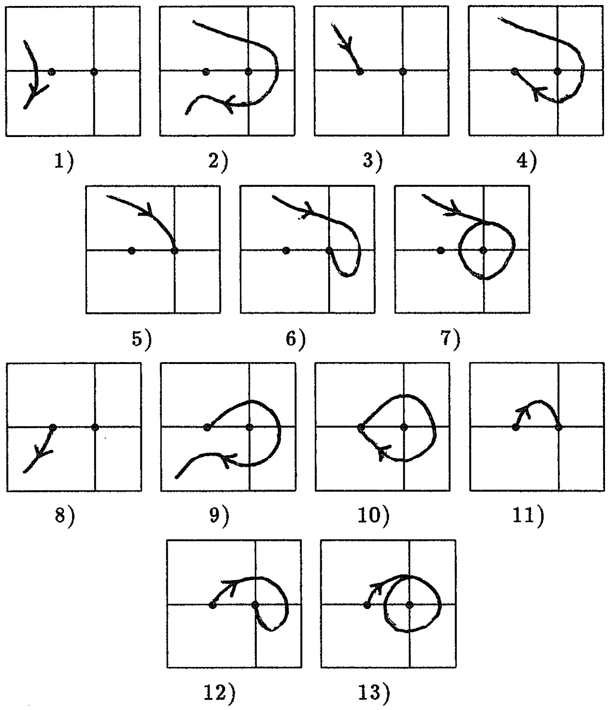
Trajectory gathering then proceeds progressively. Initially, paths emerging from points at infinity and fixed points are traced. This results in the paths shown in Figure 2b. Note that topologically equivalent paths are abstracted together. The cycle associated with trajectories 7 and 13 is then refined to extract its possible trends. It is found to be inward spiraling, and is consistent with trajectories 7 and 13. Further processing of trajectories 7 and 13 produce trajectories that spiral into the sink in various manners.

Subsequently, when analyzing trajectories for spiral-nodal bundles, spiraling trajectories associated with 7, together with 5 and 6, are bundled. Also bundled are spiraling trajectories associated with 13, together with 11 and 12.

[5]A $U^+_{(a,b)}$ function is a QSIM [Kuipers 86] modeling primitive. Intuitively speaking, it is a 'U' shaped function consisting of a monotonically decreasing left segment and a monotonically increasing right segment, with (a, b) the bottom point.



a) Envisionment graph of the Lienard equation in the phase plane.



b) Trajectories from initial gathering. Trajectories 7 and 13 are cyclic and incomplete. Trajectory 10 is a homoclinic orbit.

Figure 2: Intermediate results from applying QPORTRAIT to a QDE generalization of a Lienard equation.

Trajectory 10 is a separatrix connecting a saddle to itself (a homoclinic orbit). It is a potential asymptotic limit, and is further processed for trajectories emerging from or terminating on it. Subsequent checking for consistent behavior refinements of trajectories, however, finds trajectory 10 to be inconsistent (violating energy constraints). Trajectory 10 and its associated trajectories are therefore eliminated. Trajectory 9 is also found to violate energy constraints and eliminated.

Trajectories resulting from gathering are 1 through 4, 8, and the two bundles. Of these, 3, 4 , 8 and the bundle associated with 13 are separatrices. Composing separatrix set, then phase portrait, give the result in Figure 1b.

Discussions

While some phase portraits produced by QPORTRAIT may be spurious, and some may contain spurious trajectories, the set of portraits that remain consistent after spurious trajectories are removed is guaranteed to capture all real portraits of systems consistent with the given QDE. We have applied QPORTRAIT to obtain tractable results for a set of nontrivial examples to offer reasonable coverage of possible asymptotic limits of systems in our domain. Included are a Lienard equation, a van der Pol equation, an undamped pendulum, and a predator-prey system.

Guarantee of Coverage

Guaranteed coverage follows from the guarantees of the individual steps. First, qualitative simulation is guaranteed to predict all qualitatively distinct solutions. Second, possible asymptotic limits of trajectories are exhaustively identified for systems in our domain. Third, possible flows between asymptotic limits are exhaustively traced, eliminating only provably inconsistent flows. Fourth, in abstracting away uninteresting qualitative distinctions, asymptotic limits and flows are preserved. Fifth, all possible phase portrait compositions are exhaustively explored, eliminating only provably inconsistent compositions. Thus, given a QDE, QPORTRAIT is guaranteed to produce all qualitatively distinct phase portraits of it.

Dependencies and Limitations[6]

While QPORTRAIT is dependent on its supporting techniques, the dependency is in terms of tractability[7]. In other words, improvement in performance of the supporting techniques gives more tractable results, and converse otherwise. Guarantee of coverage, however, is preserved regardless of the performance of the supporting techniques, though the guarantee becomes increasingly less useful as results become increasingly less tractable.

Although QPORTRAIT is able to produce tractable results for the examples we've attempted, it would not be difficult to come up with examples where intractability would result. No general characterization relating system property to the potential for intractability has been developed. Nevertheless, knowledge of system fixed points helps produce tracable results, such as when fixed points are at landmark values for the phase variables.

QPORTRAIT's applicability is limited to autonomous two-dimensional systems with nondegenerate fixed points. Extending QPORTRAIT to apply to systems with degenerate fixed points would require incorporating knowledge of asymptotic limits of such systems. While nonautonomous systems can be transformed into equivalent autonomous systems, systems of higher dimensions will result. Extending QPORTRAIT to higher dimensional systems will be difficult, largely because the qualitative non-intersection constraint [Lee & Kuipers 88] may not apply generally. Furthermore, trajectory flows and their asymptotic limits have more complicated structures in higher-dimensional systems, and are hard to characterize exhaustively.

Related Work

Various numeric methods based approaches to reason in the phase space have recently emerged. These include the work of Nishida et al, Sacks, Yip, and Zhao. They work with exactly specified system instances to produce approximate solutions, and are able to produce qualitative conclusions from underlying numerical results. Although each approach iterates in an attempt to capture all essential qualitative features, none guarantees coverage.

An early attempt to use qualitative simulation to construct phase portraits is the work of [Chiu 88]. Chiu was able to use the few available qualitative simulation techniques to perform complete analysis of various systems. Using his work as our foundation, we are able to take advantage of more recently developed techniques to perform more sophisticated reasoning, and incorporate sufficient knowledge of dynamical systems to handle a significant class of systems.

Conclusion

We have developed a qualitative simulation based method to construct phase portraits of autonomous two-dimensional differential equations with nondegenerate fixed points. It has been implemented in the QPORTRAIT program. It has the attractive property that it is guaranteed to capture the essential qualitative features of all real phase portraits of systems

[6] Aspects concerning construction of QDE, determination of nondegeneracy of fixed points, and system bifurcation have been discussed when describing steps of QPORTRAIT, and will not be repeated here.

[7] The problem with tractability can be due to an intractable number of spurious predictions, or an intractable number of overly detailed distinctions, or both. Refer to [Lee 93] for further discussion.

consistent with an incomplete state of knowledge (a QDE). This complements the ability of numeric methods based approaches to produce good approximate results for particular system instances.

While the potential for intractable results remain, we have demonstrated that QPORTRAIT is able to produce tractable results for nontrivial systems. In particular, results will be tractable when fixed points of the system are at landmark values for the phase variables.

Extending our approach to higher-dimensional systems will be hard, and will be a very significant contribution. It will need to proceed in smaller steps (covering a smaller class of systems at a time) due to the more complicated phase space structures of higher-dimensional systems. Integration with numeric methods to combine the power of both approaches appears to be a particularly attractive line of future work.

Despite a concern (notably in [Sacks & Doyle 92a] and [Sacks & Doyle 92b]) that qualitative simulation methods may not be useful for scientific and engineering reasoning, our work represents a significant steps towards automated reasoning about differential equations, which are important for scientists and engineers. Furthermore, our work is a demonstration that a dynamical systems (phase space) perspective can give a tractable overview of a qualitative simulation problem.

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