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Algebraic Semantics for Cumulative Inference Operations

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Abstract

In this paper we propose preferential matrix semantics for nonmonotonic inference systems and show how this algebraic framework can be used in methodological studies of cumulative inference operations.

Introduction

The notion of a cumulative inference operation arose as a result of formal studies of properties of nonmonotonic inference systems, more specifically, as a result of the search for desired and natural formal properties of such inference systems. In this paper we propose an algebraic semantics for cumulative inference systems and show how this new semantic framework can be used for the methodological studies of nonmonotonic reasoning. The point of departure for our presentation are the studies of nonmonotonic inference systems undertaken in (Brown & Shoham 1988, Gabbay 1985, Kraus, Lehmann, & Magidor 1990, Makinson 1988). All these works share a common preferential model-theoretic view on semantics. In (Makinson 1988) and (Makinson 1989) this unified semantic framework assumes the form of the theory of preferential model structures. If $\mathcal{L}$ is the language of an inference system, then a preferential model structure for $\mathcal{L}$ is a triple $M = <\mathcal{U}, \models, \prec>$, where $\mathcal{U}$ is a nonempty set the elements of which are called models, $\prec$ is a binary relation on $\mathcal{U}$, called the preference relation of M, and $\models$ is a binary relation between models in $\mathcal{U}$ and formulas of $\mathcal{L}$ called the satisfaction relation of M. No properties of $\mathcal{U}$, of the satisfaction relation $\models$, nor of the preference relation $\prec$ are assumed. Makinson shows that cumulative inference systems are exactly those defined by the class of (stoppered) preferential model structures.

The study of general properties of cumulative inference systems can be based on a less general and more structured notion of a model structure. The key feature of our semantic proposal is the truth-functional interpretation of logical connectives, the idea well-developed in the context of logical calculi which can also be exploited in the studies of nonmonotonic reasoning. We define a preferential model, or as it is called in this paper, a *preferential matrix*, as an algebra of truth-values augmented with a family $\mathcal{D}$ of sets of designated truth-values. We model a desired degree of nonmonotonicity by selecting an appropriate preference relation on $\mathcal{D}$. Preferential matrices have the same semantic scope as Makinson's preferential model structures. Evident similarities between 'classical' logical matrices and preferential matrices provide an access to reach algebraic techniques available for methodological studies of deductive proof systems. In this context, the present paper examines a list of properties of nonmonotonic inference systems, starting with characterization of *cumulativity* and *loop-cumulativity* in terms of preferential matrices. We introduce a handy notion of the *monotone base* of an inference system and study the *distributivity* property in terms of this notion. Finally, we look at the *consistency preservation* property in the context of finding automated theorem proving methods for cumulative inference systems. We give a criterion for such systems to have a refutationally equivalent automated proof system based on the resolution rule.

In this paper we study inference systems on propositional level only. It is assumed that the reader is familiar with (Gabbay 1985, Kraus, Lehmann, & Magidor 1990, Makinson 1988). The familiarity with (Brown & Shoham 1988, Makinson 1989) and the basic facts on logical matrices, as presented in (Wójcicki 1988), is an asset.

Preferential Matrices

We begin this section with a brief description of the class

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of cumulative inference systems. To avoid a lengthy exposition of facts available elsewhere, this description is just a list of definitions with rather scarce commentaries. The reader may refer to (Gabbay 1985, Kraus, Lehmann, & Magidor 1990, Makinson 1988, Makinson 1989, Wójcicki 1988) where all these definitions are fully motivated and discussed.

A *propositional language* $\mathcal{L}$ is defined in the usual way in terms of a finite set $\{f_0, ... f_k\}$ of logical connectives and a countable infinite set of propositional variables. By L we denote the set of all well-formed formulas of $\mathcal{L}$. From algebraic point of view, $\mathcal{L}$ is an algebra $< L, f_0, ..., f_k >$ of formulas while logical substitutions are simply endomorphisms of $\mathcal{L}$. [1]

Following (Makinson 1988), we say that an operation $C : 2^L \longrightarrow 2^L$ is a *cumulative inference operation* if it satisfies the following two conditions: for all $X, Y \subseteq L$,

(c1) $X \subseteq C(X)$, (*inclusion*)

(c2) $X \subseteq Y \subseteq C(X)$ implies $C(Y) = C(X)$ (*cumulativity*).

These (or equivalent) conditions were discussed in depth in (Gabbay 1985, Kraus, Lehmann, & Magidor 1990, Makinson 1988, Makinson 1989). Let us note, that (c1) and (c2) imply:

(c3) $C(C(X)) = C(X)$ (*indempotence*).

If $\alpha \in L$ and $X \subseteq L$, then we read '$\alpha \in C(X)$' as 'X entails α'. An inference operation C is a consequence operation if, in addition to (c1) and (c3), it satisfies the following condition: for every $X, Y \subseteq L$,

(c4) $X \subseteq Y$ implies $C(X) \subseteq C(Y)$ (*monotonicity*).

Every system $< \mathcal{L}, C >$, where C is a cumulative inference operation on $\mathcal{L}$ is called an *inference system*. If C is a consequence operation, then $< \mathcal{L}, C >$ is called a *logic*. If C_0, C_1 are two inference operations on $\mathcal{L}$, then we shall write $C_0 \leq C_1$ if for every $X \subseteq L, C_0(X) \subseteq C_1(X)$.

We begin our voyage towards the notion of a preferential matrix by analyzing the classical notion of a logical matrix (cf. Wójcicki 1988). Let $\mathcal{L} =< L, f_0, ..., f_k >$ be an arbitrary language fixed for the rest of this paper. A *logical matrix* is a pair $M =< \mathcal{A}, \mathcal{D} >$, where $\mathcal{A} =< A, F_0, ..., F_k >$ is an algebra of truth-values, with the set A of truth-values and with the operations $F_0, ..., F_k$ serving as interpretations of the connectives $f_0, ..., f_k$, respectively. [2] The role of $\mathcal{A}$ is to provide the interpretation of logical connectives and to define the space of truth-values – the possible meanings of formulas of $\mathcal{L}$. $\mathcal{D}$ is a family of sets of truth-values (i.e. subsets of A). We consider every $d \in \mathcal{D}$ a *set of designated truth-values*. Interpretations of formulas of $\mathcal{L}$ are defined in terms of *valuations* of $\mathcal{L}$ into M, which are simply homomorphisms of $\mathcal{L}$ into the algebra $\mathcal{A}$ of truth-values. Every logical matrix M defines the consequence operation Cn_M in the following way: for every $X \cup \{\alpha\} \subseteq L$,

(M) $\alpha \in Cn_M(X)$ *iff* for every valuation h and every $d \in \mathcal{D}, h(X) \subseteq d$ implies $h(\alpha) \in d$.

The matrix consequence operation Cn_M is always *structural*, i.e. it satisfies the following property: for every $X \subseteq L$ and every substitution e,

(c5) $e(Cn_M(X)) \subseteq Cn_M(e(X))$.

Structurality allows us to regard an entailment $\alpha \in C(X)$ as the schema representing all entailments of the form $e(\alpha) \in C(e(X))$, where e is any substitution. Moreover, the set of tautologies of a structural inference operation is closed under arbitrary substitutions. In fact, the majority of propositional logics considered in the literature, in addition to being monotonic, are structural. Nonmonotonic formalisms depart not only from the monotonicity, but frequently from the structurality as well. One way of extending matrix semantics to cover all consequence operations (not just structural) is described in (Piochi 1983) and (Stachniak 1988). The key idea is to base the semantic entailment on a set of 'admissible valuations', i.e., to consider *generalized matrices* of the form $< \mathcal{A}, \mathcal{D}, \mathcal{H} >$, where $\mathcal{A}$ and $\mathcal{D}$ are as before, and $\mathcal{H}$ is a subset of the set of all valuations of $\mathcal{L}$ into $\mathcal{A}$. In this semantic framework, every consequence operation can be defined by a generalized matrix. Let us note that in (Kraus, Lehmann, & Magidor 1990) a similar idea (of restricting the set of possible interpretations) is used to go beyond structural nonmonotonic inference systems. Our main problem with semantic modeling of cumulative inference systems, however, is not structurality but monotonicity. To get over this problem, we employ the idea of a *preference relation* so successfully used in preferential model-theoretic semantics discussed in (Brown & Shoham 1988, Kraus, Lehmann, & Magidor 1990, Makinson 1988, Makinson 1989). We call a system

$$M =< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset >,$$

a *preferential matrix* of $\mathcal{L}$ if $\mathcal{A}, \mathcal{D}$, and $\mathcal{H}$ are as described earlier, and $\sqsubset$ is a binary relation on $\mathcal{D}$. We call $\sqsubset$ the *preference relation* of M and for every pair $d_0, d_1 \in \mathcal{D}$ we read '$d_0 \sqsubset d_1$' as 'd_0 is preferred over d_1'.

EXAMPLE 2.0: Let $\mathcal{L}$ be a language with one binary connective $\vee$, one unary connective f, and two logical

[1] In fact, $\mathcal{L}$ is an absolutely free algebra generated by propositional variables of $\mathcal{L}$.

[2] We assume that $\mathcal{L}$ and $\mathcal{A}$, as algebras, are of the same similarity type.

constants **3** and **4**. The preferential matrix M we define in this example is rather artificial; however, it is designed to provide simple illustrations of some of the properties of inference operations discussed in this paper. The truth-values of M are $0,1,2,3$. The constants **3** and **4** are interpreted as the truth-values 3 and 4, respectively, while $\vee$ and f are interpreted as the operations V and F defined in the following tables:

V	0	1	2	3
0	0	3	2	1
1	0	3	1	3
2	0	3	2	2
3	1	3	2	1

p	$F(p)$
0	3
1	2
2	1
3	0

The family $\mathcal{D}$ of designated truth-values has three sets: $\{0,2,3\},\{0,1,3\}$, and $\{1,2,3\}$. The preference relation of M is defined by: $\{0,1,3\} \sqsubset \{0,2,3\}$ and $\{1,2,3\} \sqsubset \{0,2,3\}$. Finally, the set $\mathcal{H}$ consists of all valuations of $\mathcal{L}$ into $\mathcal{A}$. □

Let $M =< \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$ be a preferential matrix. For every set $X \subseteq L$, every set d of designated truth-values of M, and every valuation $h \in \mathcal{H}$, we shall write $Sat_M(h,X,d)$ iff $h(X) \subseteq d$ and for every $d' \sqsubset d, h(X) \not\subseteq d'$. Intuitively, '$Sat_M(h,X,d)$' means that d is a most preferred set of designated truth-values containing $h(X)$. With M we associate the inference operation C_M on $\mathcal{L}$ by rewriting the definition (M) in the following way: for every $X \cup \{\alpha\} \subseteq L$,

$\alpha \in C_M(X)$ *iff* for every $h \in \mathcal{H}$ and every $d \in \mathcal{D}$, $Sat_M(h,X,d)$ implies $h(\alpha) \in d$.

One of the conceptual distinctions between preferential model structures of Makinson and preferential matrices is the fact that in our approach the preference relation does not 'work' on models but on sets of designated truth-values – components of models. In the definition of the predicate $Sat_M(h,X,d)$, we search $\mathcal{D}$ for a minimal d (with respect to the preference relation) while keeping h and the algebra $\mathcal{A}$ of truth-values fixed. However, every preferential matrix $< \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$ can be 'decomposed' into a preferential model structure $\mathcal{M} = < \mathcal{U},\models,\prec>$, where $\mathcal{U} = \{< \mathcal{A},h,d >: h \in \mathcal{H}, d \in \mathcal{D}\}$, and $< \mathcal{A},h_0,d_0 > \prec < \mathcal{A},h_1,d_1 >$ if and only if $h_0 = h_1$ and $d_0 \sqsubset d_1$. The satisfaction relation $\models$ is defined by the equivalence:

$$< \mathcal{A},h,d > \models \alpha \quad \textit{iff} \quad h(\alpha) \in d.$$

Hence, preferential matrices can be considered special cases of preferential model structures. As we shall see shortly, for methodological studies of cumulative inference systems, preferential matrices are just what we need.

We call a preferential matrix $M =< \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$ *stoppered* iff for every set A of truth-values of M, the set $\mathcal{D}_A = \{d \in \mathcal{D} : A \subseteq d\}$ is empty or has the smallest element, i.e., there exists $d_A \in \mathcal{D}_A$ such that for every $d \in \mathcal{D}_A, d \neq d_A$ implies $d_A \sqsubset d$; moreover, for no $d \in \mathcal{D}_A, d \sqsubset d_A$ is true. The notion of a stoppered matrix is a counterpart of a stoppered preferential model structure (cf. Makinson 1988, Makinson 1989): a model structure $< \mathcal{U},\models,\prec>$ is said to be *stoppered* iff for every set A of propositions and every $m \in \mathcal{U}$, if $m \models A$, then there is a minimal $n \in \mathcal{U}$ (minimal with respect to $\prec$) such that $n \models A$ and either $n = m$ or $n \prec m$. As it was pointed out by Makinson, this notion is

> partially metamathematical, as it refers to sets A of propositions and the satisfaction relation $\models$ as well as to the non-linguistic components $\mathcal{U}$ and $\prec$ of the model structure. There does not appear to be any exactly equivalent purely mathematical condition. (Makinson 1989)

In contrast to this situation, the notion of a stoppered preferential matrix is defined in purely set-theoretic terms. Let us also note, that if a preferential matrix M is stoppered, then so is the preferential model structure $\mathcal{M}$ defined as in the previous paragraph.

THEOREM 2.1: If $M =< \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$ is a preferential stoppered matrix, then C_M is cumulative inference operation. Moreover, if $\mathcal{H}$ is closed under the composition with all substitutions of $\mathcal{L}$, then C_M is structural.

THEOREM 2.2: For every cumulative inference operation C there is a preferential stoppered matrix $M = < \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$ such that $C = C_M$. Moreover, if C is structural then $\mathcal{H}$ can be assumed to consists of all valuations.

Theorems 2.1 and 2.2 give us a representation theorem for cumulative inference systems in terms of preferential matrices. One of the matrices that satisfy Theorem 2.2 is $M_{\mathcal{L}} =< \mathcal{L},\{C(X) : X \subseteq L\},\{id\},\sqsubset>$, where id is the identity function on $\mathcal{L}$, and $C(X) \sqsubset C(Y)$ iff $C(X) \neq C(Y)$ and for some $X' \subseteq C(Y), C(X) = C(X')$. Its construction resembles that of the so-called Lindenbaum matrix for a logical system (cf. Wójcicki 1988). Henceforth, we shall call it the *Lindenbaum matrix* for C. A model structure similar to $M_{\mathcal{L}}$ is used in (Makinson 1988) to characterize the class of cumulative inference operations in terms of preferential model structures.

There are obvious connections between preferential matrices and logical matrices (for every preferential matrix $< \mathcal{A},\mathcal{D},\mathcal{H},\sqsubset>$, $< \mathcal{A},\mathcal{D} >$ is a logical matrix). Hence, one may expect to transfer some of the algebraic tools and techniques developed for logical matrices to study cumulative inference systems. In this and the following sections we will try to do just that.

Let us consider the so-called *loop principle* (cf. Kraus, Lehmann, & Magidor 1990, Makinson 1988, Makinson 1989):

(loop) $X_0 \subseteq C(X_1), X_1 \subseteq C(X_2), ..., X_{n-1} \subseteq C(X_n)$, $X_n \subseteq C(X_0)$ implies $C(X_0) = C(X_n)$.

In (Kraus, Lehmann, & Magidor 1990) and (Makinson 1989) this principle has been found the counterpart of transitivity of preference relation in stoppered preferential models. In preferential matrix semantics, loop can be related to the following property of preference relations. We say that a preferential matrix $< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset >$ is *loop-free* if and only if there is no sequence $d_0 \sqsubset d_1 \sqsubset ... \sqsubset d_n \sqsubset d_0$ of sets in $\mathcal{D}$. Following (Kraus, Lehmann, & Magidor 1990), we call every inference operation satisfying (loop) *loop-cumulative*. Our last theorem of this section characterizes the loop principle in the class of cumulative systems.

THEOREM 2.3. *Representation Theorem for Loop-Cumulative Inference Operations.* An inference operation C is loop-cumulative if and only if it is defined by a stoppered loop-free matrix.

Monotone Bases of Inference Operations

Nonmonotonic inference systems are build 'on top' or 'on the basis of' some monotonic logical systems; they depart from their deductive counterparts by giving up monotonicity for some other principles of inference. The system $\mathcal{C}$ presented in (Kraus, Lehmann, & Magidor 1990) is based on the classical logic $< \mathcal{L}_2, C_2 >$ and so are all the inference systems $< \mathcal{L}_2, C >$ such that $C_2 \leq C$. Frequently, nonmonotonic inference systems are based on non-classical logics: on Kleene's three-valued logic (cf. Doherty 1991), on modal logics (cf. McDermott 1982, Moore 1985), on constructive logic (cf. Pearce 1992), etc. This suggests the following definition:

> *the monotone base* of a cumulative inference operation C is the largest structural consequence operation C_B such that $C_B \leq C$ (the largest with respect to $\leq$).

To the best of our knowledge, no formal discussion of the monotone base of inference systems is present in the literature, although many important properties of nonmonotonic inference systems were implicitly defined in terms of this notion. Two of such properties, distributivity and consistency preservation, will be studied soon.

The fact that all structural consequence operations on $\mathcal{L}$ form a lattice under the ordering $\leq$ (cf. Wójcicki 1988) implies the following theorem:

THEOREM 3.0: Every inference operation has the unique monotone base.

In (Makinson 1989), a cumulative inference operation C is called *supraclassical* if $C_2 \leq C$, where C_2 denotes the consequence operation of the classical logic. If C is nontrivial, then the above condition is equivalent to the statement that C_2 is the monotone base of C, i.e. that $C_B = C_2$. Namely, C_2 is a maximal structural consequence operation, i.e. for every structural consequence operation $C^* \geq C_2$, $C^* = C_2$ or C^* is trivial, i.e., $C^*(X) = L$, for all $X \subseteq L$. Since C is nontrivial, by Theorem 3.0, we must have $C_B = C_2$.

Before we state our next theorem, let us introduce the following notation. If $M =< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset >$ is a preferential matrix, then C_M denotes the inference operation defined by M and Cn_M denotes the consequence operation defined by the logical matrix $< \mathcal{A}, \mathcal{D} >$.

THEOREM 3.1: For every preferential matrix M, $Cn_M \leq C_M$.

Unfortunately, Cn_M is not always the monotone base of C_M. As the next example shows, we can easily find two preferential matrices M_0 and M_1 such that $C_{M_0} = C_{M_1}$ but $Cn_{M_0} \neq Cn_{M_1}$.

EXAMPLE 3.2: Let M_0 be as in Example 1.0 and let M_1 be obtained from M_0 by the addition of $\{3\}$ to the family of designated truth-values and by making this set the maximal with respect to the preference relation. The inference operations C_{M_0} and C_{M_1} are identical while $4 \in Cn_{M_0}(\{3\}) - Cn_{M_1}(\{3\})$. The reason while $C_{M_0} = C_{M_1}$ is that the addition of the set $\{3\}$ is 'useless', i.e., it does not modify the inference engine of M_0. However, this addition modifies the 'monotone base' of M_0 represented by Cn_{M_0}. □

THEOREM 3.3: Let C be a cumulative inference operation and let M be the Lindenbaum matrix for C. Then $C_B = Cn_M$.

By Theorem 3.3, the monotone base of a cumulative inference operation C is defined by the 'logical part' of the Lindenbaum matrix for C. If C is introduced by an arbitrary matrix M, then frequently $C_B = Cn_M$, provided that M has no 'useless' sets of designated truth-values. The details are as follows. Let $M =< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset >$ be a preferential matrix. We call a set $d \in \mathcal{D}$ *useless* if for some $d^* \in \mathcal{D}, d^* \sqsubset d$ and $d \subseteq d^*$. The next theorem shows that all the useless sets can be safely eliminated.

THEOREM 3.4: Let M be a preferential matrix and let N be the matrix obtained from M by removing all useless sets. Then $C_M = C_N$.

THEOREM 3.5: Let C be a cumulative inference operation defined by a matrix $M =< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset>$ without useless sets. If $\mathcal{H}$ contains a valuation of $\mathcal{L}$ onto $\mathcal{A}$, then $C_B = Cn_M$.

Theorems 3.4 and 3.5, when applied to a preferential matrix M with the full set of valuations, say that the monotone base of C_M is defined by M with all the useless sets removed.

As we have settled the problem of semantic definition of monotone bases, let us turn to the problem of an interplay between cumulative inference operations and their monotone bases. In (Makinson 1988, Makinson 1989) it is shown that every supraclassical inference system C defined by a *classical* preferential model structure (i.e. the satisfaction relation preserves the intended meanings of classical logical connectives) satisfies the following principle of distributivity: for every $X, Y \subseteq L$,

(d) $C_2(X) = X$ and $C_2(Y) = Y$ implies $C(X) \cap C(Y) \subseteq C(X \cap Y)$.

Makinson's proof exploits the following property of classical models: a disjunction α is satisfied in a model M iff one of the disjuncts of α is satisfied in M. The preferential matrix counterpart of this property, called well-connectivity, is defined as follows. Let $M =< \mathcal{A}, \mathcal{D}, \mathcal{H}, \sqsubset>$ be a preferential matrix for $\mathcal{L}$ and let us suppose that $\vee$ is one of the binary connectives of $\mathcal{L}$. We say that M is *well-connected* (with respect to $\vee$) iff for every pair a, b of truth-values of M and every $d \in \mathcal{D}$:

$$a \vee b \in d \quad \textit{iff} \quad a \in d \text{ or } b \in d.$$

Well-connectivity is, in fact, a property of the logical matrix $< \mathcal{A}, \mathcal{D} >$, since its definition is independent of the preference relation of M. Moreover, if M is well-connected, then Cn_M is disjunctive with respect to $\vee$, i.e., for every $X \cup \{\alpha, \beta\} \subseteq L$, $Cn(X \cup \{\alpha \vee \beta\}) = C(X \cup \{\alpha\}) \cap C(X \cup \{\beta\})$. We say that a cumulative inference operation C is *distributive* if it satisfies (d) with C_2 replaced by C_B. Our next theorem extends Makinson's result to a larger class of cumulative operations.

THEOREM 3.6: Every cumulative inference operation defined by a well-connected preferential matrix is distributive.

Another result, first formulated and proved for supraclassical cumulative inference systems, which, when expressed in terms of monotone bases, holds for a larger class of such systems is presented in the following theorem.

THEOREM 3.7: Every distributive cumulative inference operation is loop-cumulative.

Consistency Preservation

An inference operation C satisfies the *consistency preservation property* if for every $X \subseteq L$,

$$C(X) = L \quad \textit{iff} \quad C_B(X) = L.$$

In other words, this property, when satisfied, sets the limit on how much a given inference operation C and its monotone base C_B may differ. In the context of supraclassical inference systems this property was investigated in (Makinson 1989), where the reader is referred to for examples of inference systems with this property. Let us note that:

THEOREM 4.0: Every cumulative structural inference operation satisfies the consistency preservation property.

The satisfaction of the consistency preservation principle enables the application of rich refutational automated theorem proving techniques available for logical systems, such as non-clausal resolution or signed tableau (cf. Hähnle 1991, Stachniak 1991, Stachniak 1992), to cumulative inference systems. In fact, following (Stachniak 1991), we can characterize the class of cumulative inference systems for which a refutationally equivalent resolution proof system can be found. Informally, a logic $\mathcal{P}$ is said to be a *resolution logic* if there exists a resolution-based proof system *Rs* refutationally equivalent to $\mathcal{P}$, i.e. for every finite set X of formulas, X is inconsistent in $\mathcal{P}$ if and only if X can be refuted in *Rs* (cf. Stachniak 1991, Stachniak 1992). This definition can be extended to cumulative inference systems in the following way: a cumulative inference system $< \mathcal{L}, C >$ with the consistency preservation property is said to be a *resolution inference system* if and only if the logic $< \mathcal{L}, C_B >$ is a resolution logic. In the light of this definition, the characterization of the class of resolution logics given in (Stachniak 1991) can be extended to resolution inference systems as follows:

THEOREM 4.1: Let C be a cumulative inference operation with the consistency preservation property. Then the following conditions are equivalent:

(i) $< \mathcal{L}, C >$ is a resolution inference system;

(ii) there exists a finite logical matrix M such that C and Cn_M have the same inconsistent sets;

(iii) for some integer $k \geq 0$, $L^{(k)}/\Theta$ is finite and for every finite set $X \subseteq L$,

$$C(X) = L \textit{ iff } C(e(X)) = L, \text{ for every substitution } e \text{ that maps } L \text{ into } L^{(k)}.$$

In this theorem, $L^{(k)}$ denotes the set of all formulas of L which are built up by means of the connectives of $\mathcal{L}$ and the propositional variables $p_0, ..., p_k$. Θ denotes the congruence relation of $\mathcal{L}$ defined in the following way: for every $\alpha, \beta \in L$,

$$\alpha\Theta\beta \textit{ iff } \text{for every } X \subseteq L \text{ and every } \gamma(p) \in L,$$
$$C(X \cup \{\gamma(p/\alpha)\}) = L \leftrightarrow C(X \cup \{\gamma(p/\beta)\}) = L.$$

If C is a structural cumulative inference operation whose monotone base is defined by a finite logical matrix N, then, by Theorems 4.0 and 4.1, $< \mathcal{L}, C >$ is a resolution inference system. Moreover, a resolution based automated proof system for $< \mathcal{L}, C >$ can be effectively constructed from N following, for example, the algorithm described in (Stachniak 1991).

Let us close this section with the following general remark. The application of refutational theorem proving techniques to a particular cumulative inference system $< \mathcal{L}, C >$ hinges upon the availability of an effective procedure that reduces the problem of validation of an inference to the problem of inconsistency checking. What is required is an algorithm which for every finite set X of formulas and every formula α constructs a finite set X_α such that the following *reduction principle* holds:

$$\alpha \in C(X) \quad \textit{iff} \quad C(X_\alpha) = L.$$

For example, in the case of the classical propositional logic we can just put $X_\alpha = X \cup \{\neg\alpha\}$. The reduction principle, however, is not universally available to all cumulative inference systems, as it is not available to all logics.

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