The Complexity of Propositional Default Logics

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Abstract

We characterize the complexity of several typical problems in propositional default logics. In particular, we examine the complexity of extension membership, extension existence, and extension entailment problems. We show that the extension existence problem is Σ_2^p complete, even for semi-normal theories, and that the extension membership and entailment problems are Σ_2^p complete and Π_2^p complete respectively, even when restricted to normal default theories. These results contribute to our understanding of the computational relationship between propositional default logics and other formalisms for nonmonotonic reasoning, e.g., autoepistemic logic and McDermott and Doyle's NML, as well as their relationship to problems outside the realm of nonmonotonic reasoning.

Introduction

Almost every activity that one undertakes involves reasoning and acting based on incomplete information. You are reading these words under the assumption that they will say something about the topic described in the abstract above, although you don't know this. Perhaps the next few pages are blank. There are many ways that your plan to continue reading could be thwarted, yet you probably haven't thought about them, nor about how you might cope with them. Normally, this incompleteness doesn't even play a role in one's conscious reasoning process.

Much of artificial intelligence research involves developing useful models of how one might emulate on computers this 'common-sense' reasoning in the presence of incomplete information that people do as a matter of course.

There have been a number of attempts at developing such models, both ad hoc and formal. Researchers argue that traditional predicate logics, developed for reasoning about mathematics, are inadequate as a formal framework for such research in that they are inherently *monotonic*: if one can derive a conclusion from a set of formulae then that same conclusion can also be derived from every superset of those formulae. They argue that people simply don't reason this way: we are constantly making assumptions about the world in light of incomplete information, acting on those assumptions, then revising our beliefs as further information becomes available (see [McCarthy 1977] or [Minsky 1975], for instance). Many researchers have proposed modifications of traditional logic to model the ability to revise conclusions in the presence of additional information (see, for instance, [McCarthy 1986], [Moore 1983], [Poole 1986]). Such logics are called *nonmonotonic*. Informally, the common idea in all these approaches is that one may want to be able to "jump to conclusions" that might have to be retracted later.

A detailed discussion of nonmonotonic logics is outside the scope of this paper; a good introduction to the topic can be found in [Etherington 1988], and a number of the most important papers in the field have been collected in [Ginsberg 1987], which also provides some good introductory material.

One of the most prominent of the formal approaches to nonmonotonic reasoning was proposed by Reiter ([Reiter 1980]). Reiter's approach is based on *default rules*, which are used to model decisions made in prototypical situations when specific or complete information is lacking. Reiter's *default logic* is an extension of first order logic that allows the specification of default rules, which we will summarize shortly. Unfortunately, the decision problem for Reiter's default logic is highly intractable in that it relies heavily on consistency checking for processing default rules, and is thus not even semi-decidable (this is not a weakness of Reiter's logic alone; it is common to most nonmonotonic logics). This precludes the practical use of Reiter's full default logic in most situations.

In earlier work of this author's [Stillman 1990a; Stillman 1990b] and Kautz and Selman's [Kautz and Selman 1989], syntactically restricted propositional default theories were investigated in attempts to identifying regions of tractability in default reasoning. The work was motivated by the need to reason about relatively large propositional knowledge bases in which the default structures may be quite simple. Recent research involving inheritance networks with exceptions is particularly relevant, and is explored in depth

in [Touretzky 1986] and in Chapter 4 of [Etherington 1988], where the close relationship between default logic and inheritance networks with exceptions is explored. A partial order of syntactic restrictions is described in [Kautz and Selman 1989], together with discussion of the complexity of several problems over this partial order when the propositional theory is restricted to consisting of a set of literals. Their results showed that while very simple theories may be intractable, some syntactic restrictions result in polynomial-time tests for determining whether certain properties hold. In particular, they showed that one can decide in polynomial time whether there exists an *extension* that contains a given literal when the default rules are restricted to a class they called *Horn* default rules. They suggested that the ability to combine such default theories with non-default propositional Horn theories would be particularly useful, but left open the question of whether the membership problem (i.e., determining whether there exists an extension of a given default theory containing a specified literal) for such a combination of theories is tractable. In [Stillman 1990a], we showed that a restriction of this problem is NP-complete, and presented several related results. Our investigation in [Stillman 1990b] resulted in defining a richer hierarchy of default rules than that considered by Kautz and Selman, most of which result from disallowing any prerequisites in rules. This corresponds to introducing a "context-free" element to the reasoning, and seems to constitute the most simple type of default rule that is not completely trivial. The work described in [Stillman 1990b] considered very tight restrictions on the expressiveness of default rules as well as the underlying propositional theory. Unfortunately, our results showed that even under these restrictions, membership problems almost invariably remain intractable. These results suggested that if practical default reasoning systems are desired, one must either consider extremely restricted expressiveness or work to identify subcases of otherwise intractable classes that yield feasible complexity.

This paper contains the results of an examination of the general complexity of membership, entailment, and existence problems for arbitrary propositional default logics. Although it is clear that such problems cannot be tractable (since they trivially contain the tautology problem for propositional calculus), it is not obvious exactly how hard they are. The main results we present are that these problems are indeed higher in the polynomial hierarchy than NP or co-NP, placing them at the second level of the hierarchy. While the impact of classifying a problem at the second level of the hierarchy is not yet fully understood (similarly, we don't *know* that showing a problem to NP-complete means that it can't be solved efficiently, just that it's in the same set as a number of other problems that we don't know how to solve efficiently), this may indicate that these problems are strictly harder than those in NP (It is important to note, however, that $\mathbf{P} = \mathbf{NP}$ if and only if $\mathbf{P} = \mathbf{PH}$, the polynomial hierarchy, i.e., if and only if the polynomial hierarchy collapses to $\mathbf{P}$.), although all of the problems in the polynomial hierarchy can be solved in single exponential time. From a practical standpoint, a more complete understanding of just where the important decision problems in propositional default logic lie in the polynomial hierarchy may help us understand how to develop better deterministic algorithms for these problems by enabling us to exploit prior results about other problems that lie in the same classes.

The remainder of this paper is organized as follows: we begin with a brief description of Reiter's default logic, followed by a short overview of the polynomial hierarchy. We then present the main results of this paper, characterizing the complexity of several key decision problems for propositional default theories. Finally, we summarize and discuss related results and future work.

Preliminaries

Reiter's Default Logic

For a detailed discussion of Reiter's default logic the interested reader is referred to [Reiter 1980]. In this section we will simply review some of the immediately pertinent ideas. A *default theory* is a pair (D, W), where W is a set of closed well-formed formulae (wffs) in a first order language and D is a set of *default rules*. A *default rule* consists of a triple $< \alpha, \beta, \gamma >$: α is a formula called the *prerequisite*, β is a set of formulae called the *justifications*, and γ is a formula called the *conclusion*. Informally, a default rule denotes the statement "if the *prerequisite* is true, and the *justifications* are consistent with what is believed, then one may infer the *conclusion*." Default rules are written

$$\frac{\alpha : \beta}{\gamma}$$

If the conclusion of a default rule occurs in the justifications, the default rule is said to be *semi-normal*; if the conclusion is identical to the justifications the rule is said to be *normal*. A default rule is *closed* if it does not have any free occurrences of variables, and a default theory is *closed* if all of its rules are closed.

The maximally consistent sets that can follow from a default theory are called *extensions*. An extension can be thought of informally as one way of "filling in the gaps about the world." Formally, an extension E of a closed set of wffs T is defined as the fixed-point of an operator Γ, where $\Gamma(T)$ is the smallest set satisfying:

- $W \subseteq \Gamma(T)$,
- $\Gamma(T)$ is deductively closed,
- for each default $d \in D$, if the *prerequisite* is in $\Gamma(T)$, and T does *not* contain the negations of any of the *justifications*, then the *conclusion* is in $\Gamma(T)$.

Since the operator Γ is not necessarily monotonic, a default theory may not have any extensions. Normal default theories do not suffer from this, however (see [Reiter 1980]), and always have at least one extension.

In [Reiter 1980] an alternative characterization of the extensions of a default theory is provided and proved equivalent to that given above:

Theorem 1 (Reiter) *Let E be a closed set of wffs, and let $\Delta = (D, W)$ be a closed default theory. Define*

$$E_0 = W$$

and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \begin{array}{ll} \dfrac{\alpha : \beta_1, \ldots \beta_m}{\gamma} & \in D \\ \text{where } \alpha \in E_i \text{ and } \neg\beta_1, \ldots \neg\beta_m \notin E & \end{array} \right\}$$

Then E is an extension for Δ iff

$$E = \bigcup_{i=0}^{\infty} E_i$$

Properties of Default Theories

There are several important properties that may hold for a default theory. Given a default theory (D, W), perhaps together with a literal q, one might want to determine the following about its extensions:

Existence Does there exist *any* extension of (D, W)?

Membership Does there exist an extension of (D, W) that contains q? (This is called *goal-directed reasoning* by Kautz and Selman.)

Entailment Does every extension of (D, W) contain q? (This is closely related to *skeptical reasoning*, where a literal is believed if and only if it is included in all extensions.)

The Polynomial Hierarchy and Σ_k^p Complete Problems

There exist many problems for which it is unknown whether or not they can be solved efficiently (in time polynomial in the length of their input). Structural complexity theory is concerned with categorizing such problems into equivalence classes, and with determining relationships between such classes. The most well known class studied in structural complexity theory is NP; hundreds of commons problems in many areas of study have been identified as being NP-complete or NP-hard (see [Garey and Johnson 1979] for a good introduction to this topic). Much of the research in structural complexity theory is directed toward providing insight into what is perhaps the most famous problem in computer science: 'What is the effect of nondeterminism on computational complexity?' This includes the famous P = NP question.

There exist decision problems that seem to be harder than NP-complete problems, but which we have been unable to prove to be so. The *polynomial hierarchy*, formally defined in [Stockmeyer 1976] (see also [Garey and Johnson 1979] for an introduction) is a way of classifying some of these NP-hard decision problems by considering the ramifications of providing time-bounded Turing machines with oracles. Thus the polynomial hierarchy is a subrecursive analog of the Kleene arithmetical hierarchy [Rogers 1967], in which deterministic polynomial time is substituted for recursive time, and nondeterministic polynomial time is substituted for recursively enumerable time.

The classes that comprise the polynomial hierarchy are defined as the set

$$\{\Sigma_k^p, \Pi_k^p, \Delta_k^p : k \geq 0\},$$

where

$$\Sigma_0^p = \Pi_0^p = \Delta_0^p = \mathbf{P};$$

and for $k \geq 0$,

$$\Sigma_{k+1}^p = \mathbf{NP}(\Sigma_k^p),$$

$$\Pi_{k+1}^p = \mathbf{co-NP}(\Sigma_k^p),$$

$$\Delta_{k+1}^p = \mathbf{P}(\Sigma_k^p).$$

(Note in particular that $\Sigma_1^p = \mathbf{NP}$, and that $\Pi_1^p = \mathbf{co-NP}$.)

It is helpful to have a canonical complete problem for each of these classes as a starting point in proving completeness of new problems. This role is played be the problem SATISFIABILITY in the theory of NP-completeness. In [Meyer,Stockmeyer 1973], a class of problems B_k is defined for $k \geq 0$, and it is shown that B_k is complete for Σ_k^p. An instance of B_k consists of:

Input: A Boolean formula F over sets of variables $\overline{X}_1, \overline{X}_2, \ldots \overline{X}_k$.

Question: Does

$$\exists \overline{X}_1, \forall \overline{X}_2, \ldots Q_k \overline{X}_k \ \ [f(\overline{X}_1, \overline{X}_2, \ldots \overline{X}_k)] \ = 1?$$

where Q_k is $\exists$ if k is odd, $\forall$ if k is even.

Main Results

We are now ready to present the main results of this paper:

Theorem 2 *The Extension Membership Problem for propositional default theories is Σ_2^p-complete.*

Proof: (sketch) To show that the Extension Membership Problem (**EMP**) is Σ_2^p-hard we reduce B_2 to EMP as follows:
Let

$$\exists(\overline{X})\forall(\overline{Y})[f(\overline{X}, \overline{Y})]$$

be an instance of B_2, where $(\overline{X})$ and $(\overline{Y})$ each denote a set of boolean variables. The instance of EMP we will construct has as its default theory $\Delta = (D, W)$ with

$W = \phi$, and the set of default rules D is composed of the following. For each $x_i \in \overline{X}$ we introduce a new variable e_i and two default rules:

$$\frac{: x_i \wedge e_i}{x_i \wedge e_i} \qquad \frac{: \neg x_i \wedge e_i}{\neg x_i \wedge e_i}$$

Having done this, we introduce a new variable q and add the default rule

$$\frac{e_1 \wedge e_2 \wedge \ldots e_{|\overline{X}|} \wedge f(\overline{X}, \overline{Y}) : q}{q}$$

This completes the construction. Intuitively, the new variables e_i act as *enablers* (the use of these variables is not strictly necessary, but their presence aids in understanding the proof), in that they indicate that some truth value has been chosen for the variable x_i, and the variable q is used to indicate that the formula is a tautology given the choice of assignment to $\overline{X}$.

Suppose the instance is in B_2, i.e., there does exist an assignment to those variables in $\overline{X}$ such that for all assignments to the variables in $\overline{Y}$, $f(\overline{X}, \overline{Y}) = 1$. To construct an extension E that contains the literal q, we choose an arbitrary assignment α to the variables in $\overline{X}$ such that $f(\alpha(\overline{X}), \overline{Y}) = 1$. For each $x_i \in \overline{X}$, if $\alpha(x_i) = 1$, include in E the consequences of applying the default rule

$$\frac{: x_i \wedge e_i}{x_i \wedge e_i}$$

Similarly, if if $\alpha(x_i) = 0$, include in E the consequences of applying the default rule

$$\frac{: \neg x_i \wedge e_i}{\neg x_i \wedge e_i}$$

Once this is done, each of the variables e_i is true. Furthermore, by hypothesis, $f(\alpha(\overline{X}), \overline{Y}) = 1$. Thus the default rule

$$\frac{e_1 \wedge e_2 \wedge \ldots e_m \wedge f(\overline{X}, \overline{Y}) : q}{q}$$

is applicable, allowing us to include q in E. It is easy to see that the result of applying the specified defaults produces an extension that contains q.

Conversely, if there exists an extension E containing q, we maintain that the instance of B_2 is satisfiable. The only way that q can be included in E is if the default rule

$$\frac{e_1 \wedge e_2 \wedge \ldots e_m \wedge f(\overline{X}, \overline{Y}) : q}{q}$$

has been applied. Thus each of the e_i variables is also in E, signifying that for each x_i either x_i or $\neg x_i$ is in E, and $f(\overline{X}, \overline{Y})$ is provable in E. Since E can impose no restrictions on $\overline{Y}$, this signifies that once the choice of a truth value for each x_i via application of default rules has been made, it is the case that for all values of the variables in $\overline{Y}$, $f(\overline{X}, \overline{Y})$ is true. It is now a trivial matter to construct a satisfying assignment for the instance of B_2.

Next we must prove that EMP is in Σ_2^p. Informally, given an arbitrary propositional default theory $\Delta = (D, W)$ we will use the *NP* machine to guess the sequence of defaults to be applied in constructing an extension E that contains q, then use the *NP* oracle to verify that the chosen defaults are applicable, that no other default rules can be applied (E is an extension), and that $q \in E$. (It should be noted that we do not need to actually create the extension; it suffices to augment W with the consequences of the chosen default rules, and verify that (a) no other default rules can be applied, and (b) the set of propositional sentences thus created is consistent. The actual size of the extension may be exponential in the size of the original presentation since it contains the logical closure of a set of propositional sentences, and thus we cannot afford to express it explicitly.) It is also important to the correctness of the proof that each default rule will be chosen at most once in deriving an extension, thus guaranteeing that the guess provided by the oracle is polynomial in length with respect to the instance. This follows from the lemma below:

Lemma 3 *Let E be any extension of a propositional default theory $\Delta = (D, W)$, with $|D| = n$. Then, using the description of extensions presented in Theorem 1, $E = \bigcup_{i=0}^{n} E_i$, i.e., at most n steps are needed to produce the extension.*

Proof of the lemma is straightforward and omitted.

More formally, we can apply the following theorem, from [Meyer,Stockmeyer 1973]. (The theorem applies to Σ_k^p for arbitrary k. We have specialized it to the case where $k = 2$ for simplicity.)

Theorem 4 (Stockmeyer-Meyer) *Let Θ be a finite alphabet and $A \subset \Theta^+$. $A \in \Sigma_2^p$ iff there is a polynomial $p(n)$, an alphabet Γ, and a ternary relation $R \in P$ such that for all $x \in \Theta^+$,*

$$x \in A \quad iff \quad \exists y \forall z \; R(x, y, z)$$

where y, z range over all words in Γ^+ of length not exceeding $p(|x|)$.

Applying this theorem is somewhat involved and is omitted. □

Several related theorems and corollaries follow straightforwardly:

Theorem 5 *The Extension Entailment Problem for propositional default theories is Π_2^p-complete.*

Corollary 6 *The Extension Membership Problem for normal propositional default theories is Σ_2^p-complete.*

Corollary 7 *The Extension Entailment Problem for normal propositional default theories is Π_2^p-complete.*

Although it is known [Reiter 1980; Etherington 1988] that the extension existence problem is trivial for normal default theories, the next theorem demonstrates that relaxing this assumption introduces intractability. Specifically, we show that

Theorem 8 *The Extension Existence Problem for propositional default theories is Σ_2^p-complete.*

Proof: (sketch) The construction is similar to that provided in the proof of Theorem 2 above, with the following default rules added:

$$\frac{:q \wedge \neg a}{q}, \qquad \frac{:a \wedge \neg c}{a}, \qquad \frac{:b \wedge \neg q}{b}$$

Arguments similar to those given in the proof of Theorem 2 above demonstrate that if the instance of B_2 is true then there is a derivation that proceeds as in Theorem 2 that concludes q by applying the default rule

$$\frac{e_1 \wedge e_2 \wedge \ldots e_{|\overline{X}|} \wedge f(\overline{X}, \overline{Y}) : q}{q}$$

This may not be an extension, however, in that one can add a to the extension by applying the second default rule above. One can verify that this is an extension.

If the instance of B_2 is false, however, then the literals in the set $\{q, a, b\}$ can only be included by applying the default rules above (the key observation here is that there is no other support for q, nor for $\neg q$). One can verify that these three rules are circular, thus there does not exist any extension. This shows that the Estension Existence Problem is Σ_2^p-hard.

Demonstrating inclusion in Σ_2^p is quite similar to Theorem 2, and is omitted here for the sake of brevity. □

We have the following:

Corollary 9 *The Extension Existence Problem for semi-normal propositional default theories is Σ_2^p-complete.*

Conclusions

In this paper, we have characterized the complexity of several fundamental problems in propositional default logics, showing that even restricted versions of the extension existence problem are Σ_2^p complete, and that such restrictions of the extension membership and entailment problems are Σ_2^p complete and Π_2^p complete respectively, This work, taken together with that presented in [Kautz and Selman 1989] and [Stillman 1990b], provides an extensive categorization of the complexity of propositional default logics.

One of the most interesting areas for further research is in understanding the ramifications of these results on known formalisms for nonmonotonic reasoning. The results described above can be reproduced for the closely related questions of derivability, arguability, and fixed-point existence within propositional NML[McDermott and Doyle 1980]. This strengthens results presented in [Niemelä 1989], who showed membership, but left open questions of completeness. Furthermore, given the efficient intertranslatability between the extensions of propositional default logics and strongly grounded extensions in autoepistemic logic [Konolige 1988], similar results can be shown to hold within that paradigm. An extended version of this paper describes our results in this area. Similar results have been derived independently by Georg Gottlob [Gottlob 1992] recently; he has looked at default logics, NML, autoepistemic logic, and the closely related logic N, defined in [Marek and Truszczynski 1990]. Rutenberg [Rutenberg 1991] has derived related results for ATMS [deKleer 1986]. Researchers have just begun to classify the structural complexity of these problems. The characterization we have established thus far will serve as an aid in understanding the relationships that exist between various formalizations of nonmonotonic reasoning and in understanding the computational relationships that exist between nonmonotonic reasoning and other problems in computer science.

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